Gauss's Law

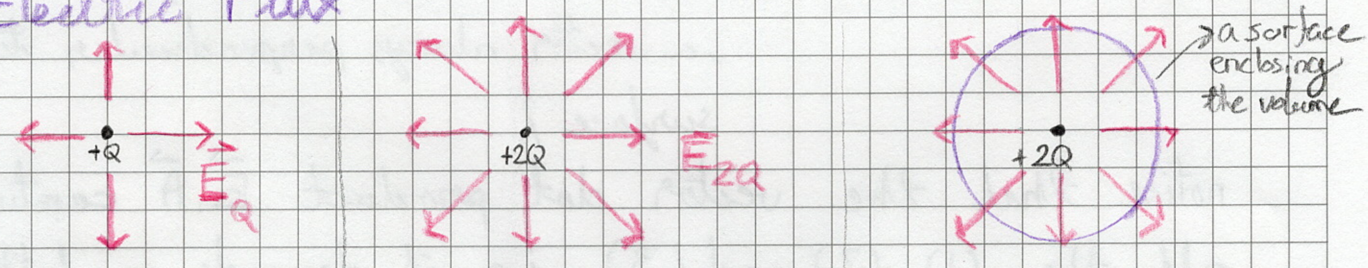
Mathematician Karl Friedrich Gauss (1777-1855)
(German)

- Illiterate mother
- Child prodigy

Gauss's law allows a more elegant approach for calculating the electric field due to a charge distrubution

→ a more general relationship between a charge distribution and the field it generates

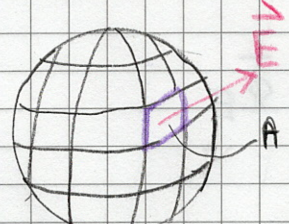
Electric Flux



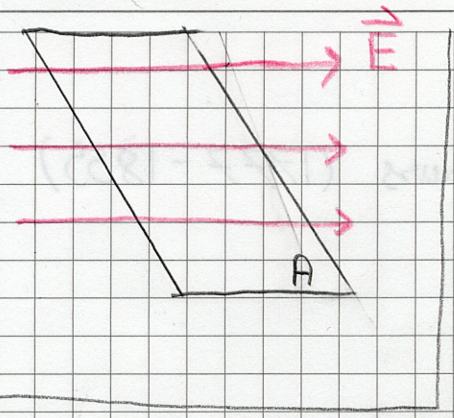
A point charge +Q generates an electric field $\vec{E}$

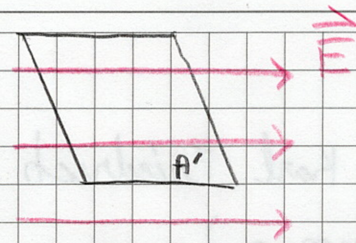
If the charge is doubled the $\vec{E}$ will double
= # of field lines will double

We can determine the intensity of $\vec{E}$ by counting all the field lines passing through a closed surface encompassing the volume charge is located

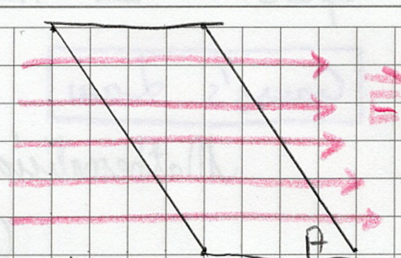


The surface can be seperated into distinct area segments and investigated piecewise

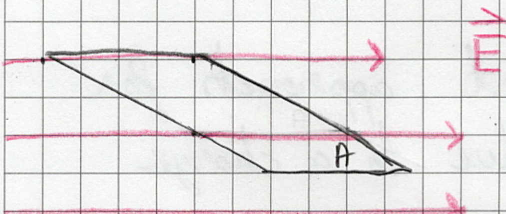




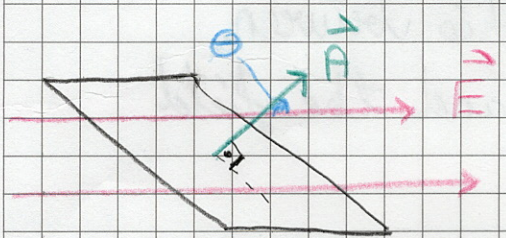
(1) when the area changes (i.e $A' < A$) number of field lines passing through it changes



(2) When the $\vec{E}$ intensifies (i.e. $\vec{E}' > \vec{E}$) number of field lines passing through the area changes



(3) when the orientation (i.e. the angle) of the area plane changes, the number of field lines passing through it changes as well.



Let's define a vector $\vec{A}$, with magnitude A (the area of the plane) and direction $\hat{n}$ (the surface normal, a vector always perpendicular to surface)

notice that the vector dot product $\vec{E}\cdot\vec{A}$ captures all the (1), (2) and (3), i.e it depends on both magnitudes of $\vec{E}$ and $\vec{A}$, and it is maximum when the plane is perpendicular to $\vec{E}$ (thus the $\hat{n}$ is parallel to $\vec{E}$)

This quantity is called the "electric flux"

$$\boxed{\Phi_E = \vec{E}\cdot\vec{A}}$$

The magnitude of $\vec{E}$ is captured by Φ_E

$$E \propto N/A_\perp \rightarrow N \propto E A_\perp = \phi_E$$

The total flux is obtained when all the patches are added together. The trick is to select small enough patches, so that the angle between $\vec{E}$ and $\vec{A}$ remains constant for each individual patch

$$A \rightarrow \Delta A_1 + \Delta A_2 + \Delta A_3 + \dots$$

When ΔA_n are small enough

- they can be considered flat
- $\vec{E}$ varies so little that it can be considered as uniform

then

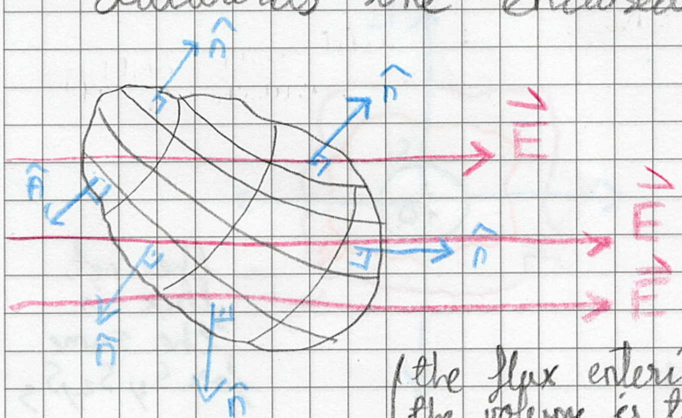
$$\phi_E = \int \vec{E} . d\vec{A}$$

When we are trying to calculate flux through a closed surface

$$\boxed{\phi_E = \oint \vec{E} . d\vec{A}}$$

surface integral

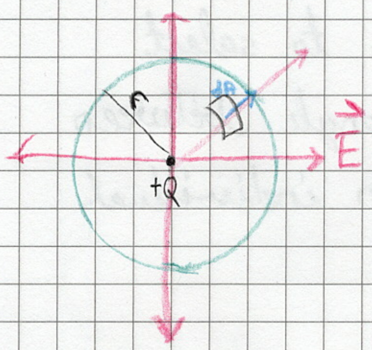
the surface normal of a closed surface points outwards the enclosed volume.



(the flux entering the volume is the same as exiting net = 0)

Thus if there is nothing that results in an $\vec{E}$ inside the enclosed volume of a closed surface, the net flux sums to zero

Gauss's Law



Now let's continue where we have left
let's count the field lines due to
a point charge using a sphere

The $\vec{E}$ due to a point charge is
spherically symmetric (i.e it does not
depend on the direction, only the distance)
on the surface of the sphere which has the point
charge at the center and has radius r

$$E = \frac{kQ}{r^2} \quad \text{(at the surface of the sphere)}$$

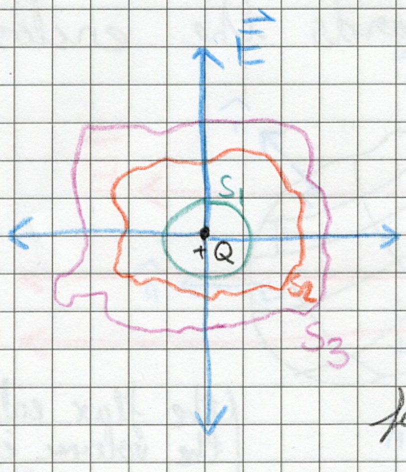
since $\vec{E}$ is spherically symmetric $\vec{E}$ and $\hat{n}$
always have 0° meaning $\vec{E}.\vec{A} = EA$

$$\Phi_E = \oint \vec{E}.d\vec{A} = \underbrace{\oint_{sphere} E\,dA = E\oint_{sphere} dA}_{\text{spherical symmetry}} = E\,4\pi r^2$$

$$E = \frac{kQ}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2} \rightarrow \Phi_E = \frac{1}{\cancel{4\pi}\epsilon_0}\frac{Q}{\cancel{r^2}}\cancel{4\pi r^2} = \frac{Q}{\epsilon_0}$$

$$\oint \vec{E}.d\vec{A} = \frac{Q}{\epsilon_0}$$

Now notice that the
$\vec{E}$ intensity is the same
no matter which surface we use
for counting the field lines
the net flux does not change



the net flux is the same for S_1, S_2, S_3

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hence,

$$\boxed{\oint E . d\vec{A} = \frac{Q_{encl}}{\epsilon_0}}$$ Gauss's law

irrespective of the surface

What about a charge destrubution?

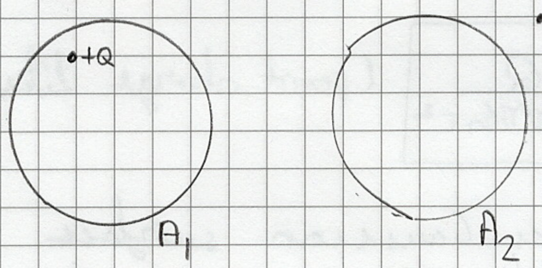
$\vec{E} = \Sigma \vec{E}_i$ (total $\vec{E}$ is superposition of each $\vec{E}_i$)

$\oint \vec{E}_i . d\vec{A} = \frac{Q_i}{\epsilon_0}$ (Gauss's law for each Q_i)

$$\rightarrow \oint \vec{E} . d\vec{A} = \oint \Sigma \vec{E}_i . d\vec{A} = \Sigma \oint \vec{E}_i . d\vec{A} = \frac{\Sigma Q_i}{\epsilon_0} = \frac{Q_{encl}}{\epsilon_0}$$

Gauss's law is more handy than Coulomb's law since it also holds for electric fields due to changing $\vec{B}$ fields etc.

ex:



What is the net flux through A_1 and A_2

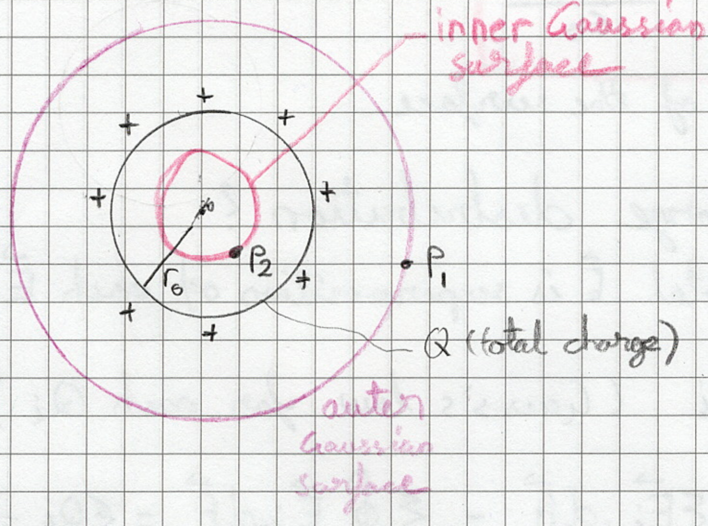
$\rightarrow A_1$: Q/ϵ_0

$\rightarrow A_2$: 0 (no charge inside

Application of Gauss's law:

- The art is in choosing the easiest Gaussian surface
- Try to use symmetry as much as possible
- Try to identify the surface that makes the integral as straightforward as possible

Example: Thin metallic spherical shell (charged)



since the shell is metallic, the charge will distribute evenly at the surface

→ spherical symmetry for $\vec{E}$ (must not depend on θ)

for $\vec{E}$ at P_1 use a sphere as Gaussian surface that has P_1 at its surface

$$\underset{\substack{\text{outer}\\\text{Gaussian}\\\text{surface}}}{\oint} \vec{E}\cdot d\vec{A} = \underbrace{\oint E\, dA}_{(\text{symmetry})} = E\oint dA = E\,4\pi r^2$$

$$E\,4\pi r^2 = \frac{Q}{\epsilon_0} \rightarrow \boxed{E = \frac{Q}{4\pi\epsilon_0 r^2}}$$ (point charge like)

for $\vec{E}$ at P_2 use a sphere as Gaussian surface that has P_2 at its surface

no charge inside

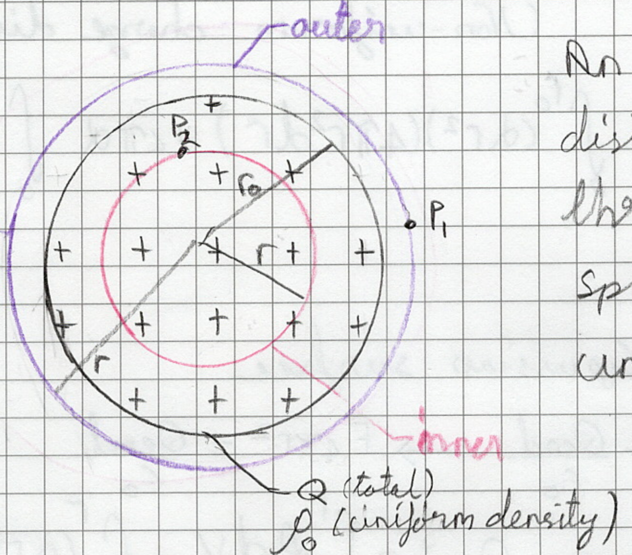
$$\oint \vec{E}\cdot d\vec{A} = E(4\pi r^2) = 0$$

not zero

must be zero

$$\boxed{E = 0}$$

Example: Solid sphere of charge



An electric charge Q is distributed uniformly throughout a non-conducting sphere.

uniform → no θ dependence
→ spherical symmetry

for the outer Gaussian surface

$$\oint \vec{E} \cdot d\vec{A} = \oint E\,dA = E \oint dA = E 4\pi r^2 = \frac{Q}{\epsilon_0}$$

$$\boxed{E_{out} = \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2}}$$

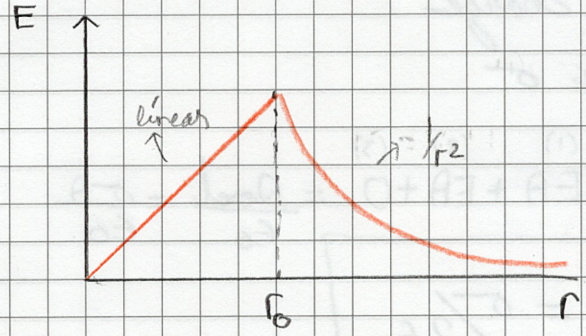
for the inner Gaussian surface

$$\oint \vec{E} \cdot d\vec{A} = E 4\pi r^2 = \frac{Q_{encl}}{\epsilon_0}$$

we need to calculate how much does this surface has inside

$$Q_{encl} = 4\pi r^3 \rho_0 = 4\pi r^3 \left(\frac{Q}{4\pi r_0^3}\right) = \frac{r^3}{r_0^3} Q$$

$$\rightarrow E 4\pi r^2 = \frac{r^3}{r_0^3} Q \rightarrow \boxed{E_{in} = \frac{1}{4\pi\epsilon_0} \frac{Q}{r_0^3} r}$$



Calculating this result wouldn't been very difficult using Coulomb's relation

Example: What if $\rho_0 \to \rho(r) = \alpha r^2$ in the previous example? (Non-uniform charge distribution)

$$Q = \int \rho \, dV = \int_0^{r_0} (\alpha r^2)(4\pi r^2 dr) = 4\pi\alpha \int_0^{r_0} r^4 dr$$

$$= \frac{4\pi\alpha}{5} r_0^5$$

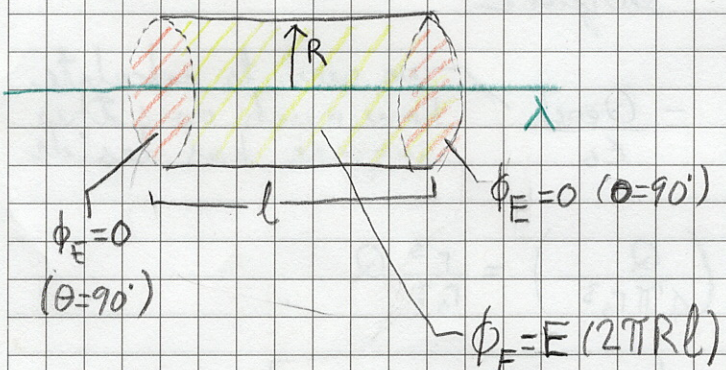
for the inner Gaussian surface

$$\oint \vec{E} \cdot d\vec{A} = \frac{Q_{encl}}{\epsilon_0} \longrightarrow E\,4\pi r^2 = \frac{Q_{encl}}{\epsilon_0}$$

$$E = \frac{Q_{encl}}{\epsilon_0 4\pi r^2} \qquad Q_{encl} = \int_0^r \rho \, dV = \int_0^r (\alpha r^2) 4\pi r^2 dr$$

$$\boxed{E = \frac{Q\, r^3}{4\pi\epsilon_0 r_0^5}} \qquad = \int_0^r \left(\frac{5Q}{4\pi r_0^5}\right) r^2 4\pi r^2 dr = Q \frac{r^5}{r_0^5}$$

Example: Long uniform line of charge

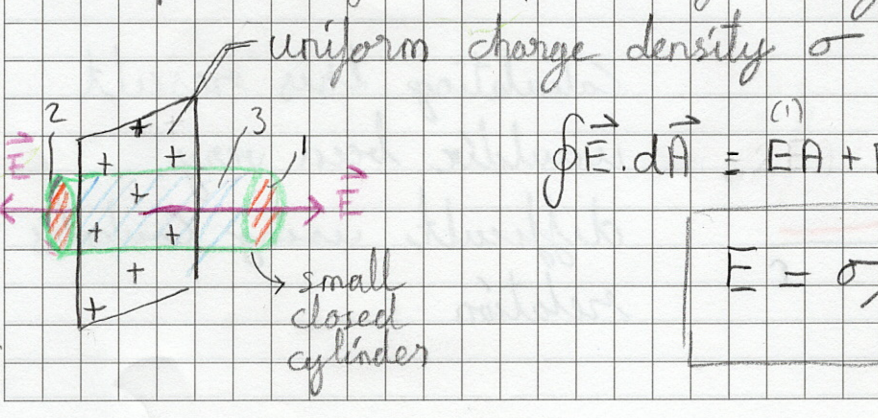


$$\oint \vec{E} \cdot d\vec{A} = 0 + 0 + E 2\pi R l = \frac{Q_{encl}}{\epsilon_0}$$

$$E 2\pi R l = \frac{\lambda l}{\epsilon_0}$$

$$\boxed{E = \frac{1}{2\pi\epsilon_0} \frac{\lambda}{R}}$$

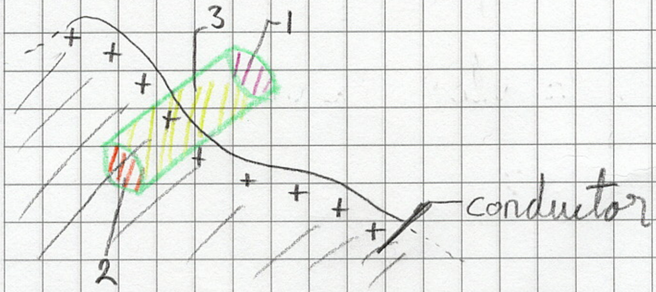
Example: Infinite plane of charge



$$\oint \vec{E} \cdot d\vec{A} = \overset{(1)}{EA} + \overset{(2)}{EA} + \overset{(3)}{0} = \frac{Q_{encl}}{\epsilon_0} = \frac{\sigma A}{\epsilon_0}$$

$$\boxed{E = \sigma / 2\epsilon_0}$$

Example : $\vec{E}$ near conducting surface



Choose an arbitrarily small Gaussian surface, such that the Area inside is flat and parallel to (1) and (2)

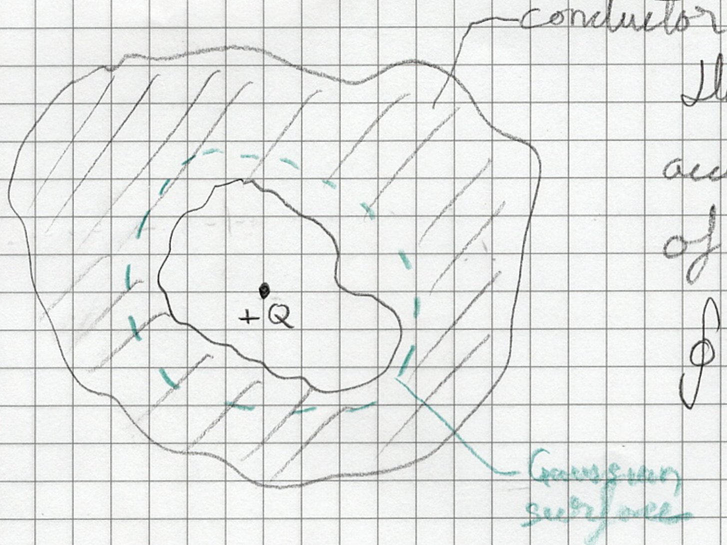
$$\oint \vec{E}\cdot d\vec{A} = \overset{(1)}{EA} + \overset{(2)}{0} + \overset{(3)}{0}$$

(2): $E=0$ inside conductor

(3): $\theta = 90$

$$\rightarrow EA = \frac{Q_{encl}}{\epsilon_0} = \frac{\sigma A}{\epsilon_0}$$

$$\boxed{E = \frac{\sigma}{\epsilon_0}}$$ (twice as before)



How much charge gets accumulated at the surface of the cavity?

$$\oint \vec{E}\cdot dA = \oint 0\, dA = 0$$

$\vec{E}$ inside conductor

$$0 = \frac{Q + Q_A}{\epsilon_0} \qquad \boxed{Q_A = -Q}$$