Rotational motion

Age of turbines

Coal → turbine → electricity
Hydro → turbine → electricity
Nuclear → turbine → electricity
Wave → turbine → electricity
Wind → turbine → electricity
Solar → no

It is very important for an engineer to understand rotational kinematics!

Let's begin our analysis with rigid objects

rigid object: An object with definitive shape that does not change

The parallels with translational motion

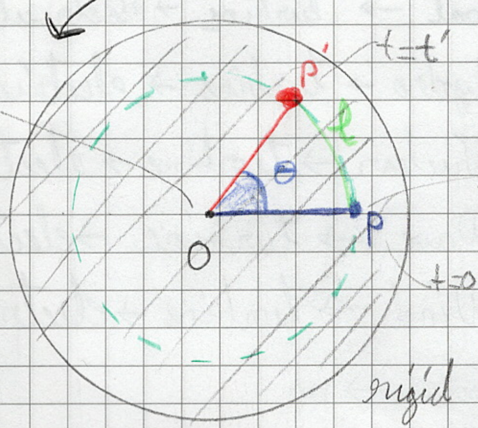
position	rotational position
↓ velocity	↓ angular velocity
↓ acceleration	↓ angular acceleration
↓ Inertial mass	↓ rotational inertia
↓ Force	↓ Torque

Angular quantities

Motion of an object = Translational motion of CM + Rotational motion about CM purely rotational motion: all points move in circles about a fixed point

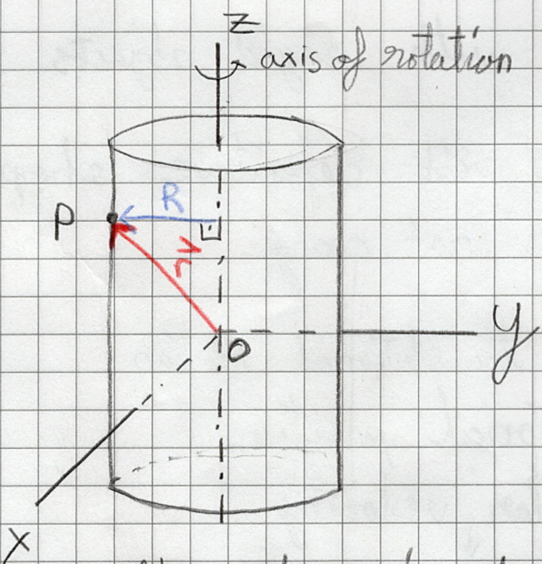
the centers of all the circles in the disk are on a line called <u>axis of rotation</u>

(passes through O, perpendicular to the paper)



→ p moves about "O" in circles

rigid disk



three dimensional rigid object

R: perpendicular distance from the axis of rotation

$\vec{r}$: position vector with respect to some origin

(note: when considering a flat and thin object (like a bicycle wheel) $R \cong |\vec{r}|$)

To indicate the angular position of an object, we specify a reference line (i.e. x axis) and find the angle to align the position line with reference line by a counter-clock-wise rotation (clockwise rotations are "−" direction, counter clockwise rotations are "+" direction, this is called a right handed choice)

Mathematics of rotational motion is much easier if angles are in radians (rad)

$$\boxed{\theta = \frac{l}{R} \qquad l = R\theta}$$

$\theta = 1$ rad $\rightarrow l = R$

when you come back to the start in a circle,

$$l = 2\pi R \rightarrow \theta = \frac{l}{R} = \frac{2\pi \cancel{R}}{\cancel{R}} = 2\pi$$

thus

$$360^\circ = 2\pi \text{ rad}$$

and

$$1 \text{ rad} = \frac{360^\circ}{2\pi} \approx 57.3^\circ$$

one revolution in radians

$$1 \text{ rev} = 360^\circ = 2\pi \text{ rad}$$

Ex: If a bird of prey's eyes have a resolving power of 3×10^{-4} rads, how small an object the bird can just distinguish when flying at a height of 100m

$$l = R\theta = (100\text{m})(3\times10^{-4}\text{rad}) = 3\times10^{-2}\text{m} = \underbrace{3\text{cm}}_{\text{small mouse}}$$

when an object rotates from θ_1 to θ_2

angular displacement : $\boxed{\Delta\theta = \theta_2 - \theta_1}$ unit rad

average angular velocity : $\boxed{\bar{\omega} = \frac{\Delta\theta}{\Delta t}}$ unit rad/s

instantaneous angular velocity : $\boxed{\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}}$

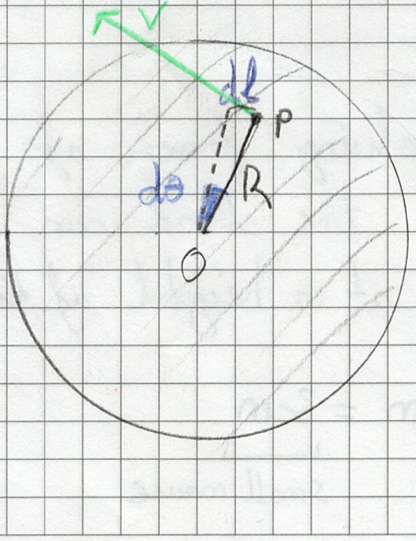
average angular acceleration : $\boxed{\bar{\alpha} = \frac{\omega_2 - \omega_1}{\Delta t} = \frac{\Delta\omega}{\Delta t}}$ unit rad/s²

instantaneous angular acceleration : $\boxed{\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt}}$

- All points in a rigid object rotate with same angular velocity (ω) (otherwise the object wouldn't be "rigid", would it?)

- Since angular velocity is the same at all points angular acceleration must be the same as well

Linear velocity and linear acceleration of a point on a rotating rigid body can be obtained from angular velocity and angular acceleration

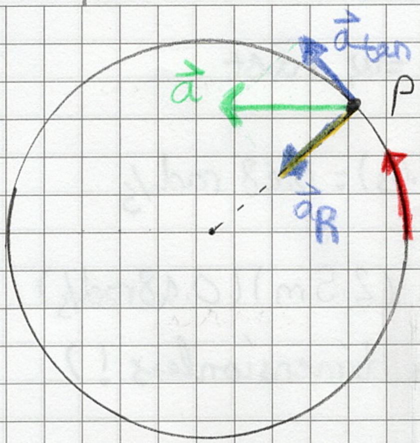


$$v = \frac{dl}{dt} = \frac{d(R\theta)}{dt} = R\frac{d\theta}{dt} = R\omega$$

$$\boxed{v = R\omega}$$

$$a_{tan} = \frac{dv}{dt} = R\frac{d\omega}{dt}$$

$$\boxed{a_{tan} = R\alpha}$$



The total linear acceleration is

$$\vec{a} = \underbrace{\vec{a}_{tan}}_{\text{tangential}} + \underbrace{\vec{a}_R}_{\text{Radial (centripetal)}}$$

$$\boxed{|\vec{a}_R| = \frac{V^2}{R} = \frac{(R\omega)^2}{R} = \omega^2 R}$$

Since this motion is repetitive, we can define a frequency (number of complete revolutions per second)

$$\boxed{f = \frac{\omega}{2\pi} \qquad \omega = 2\pi f} \quad \text{unit: Hz} \quad 1\,Hz = 1\,rev/s$$

The time required to repeat the motion once is called the period

$$\boxed{T = \frac{1}{f}}$$

ex: Angular and linear velocities and accelerations

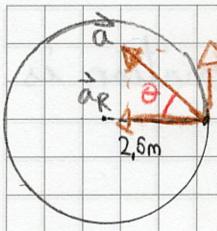
A carousel is initially at rest. At $t=0$ it is given a constant angular acceleration $\alpha = 0.060\ rad/s^2$ which increases angular velocity for 8.0s. At $t = 8.0s$ determine

a) the angular velocity

b) the linear velocity of a child located at $R = 2.5m$

c) the tangential acceleration of the child

d) the centripetal acc. e) total linear acc.



(a) $\vec{\alpha} = \dfrac{(\omega_2 - \omega_1)}{\Delta t} \rightarrow \omega_2 = \omega_1 + \vec{\alpha}\Delta t$

$$\omega_2 = 0 + (0.060\ \text{rad}/s^2)(8.0\,s) = 0.48\ \text{rad}/s$$

(b) $v(t = 8.0\,s) = R\,\omega(t = 8.0\,s) = (2.5\,m)(0.48\,\text{rad}/s)$

$v = 1.2\,m/s$ (rad is dimensionless !)

c) $a_{tan}(t = 8.0\,s) = R\alpha = (2.5\,m)(0.060\ \text{rad}/s^2) = 0.15\,m/s^2$

d) $a_R(t = 8.0\,s) = \dfrac{v^2}{R} = \dfrac{(1.2\,m/s)^2}{(2.5\,m)} = 0.58\,m/s^2$

e) a_{tan} is $\perp$ to a_R, thus

$$a = \sqrt{a_{tan}^2 + a_R^2} = 0.60\,m/s^2$$

$$\theta = \tan^{-1}\left(\frac{a_{tan}}{a_R}\right) = \tan^{-1}\left(\frac{0.15\,m/s^2}{0.58\,m/s^2}\right) = 0.25\,\text{rad}$$

$\theta \simeq 15°$

example: $\omega(t)$

A disk of radius $R = 3.0\,m$ rotates at an angular velocity $\omega = (1.6 + 1.2t)\ \text{rad}/s$, where t is in seconds. At the instant $t = 2.0\,s$ determine a) angular acceleration b) speed v, and acceleration components at a point at the edge of a disk

a)

$$\alpha = \frac{d\omega}{dt} = \frac{d}{dt}(1.6 + 1.2t)\,s^{-1} = 1.2\ \text{rad}/s^2$$

b)

$v(2.0\,s) = R\,\omega(2.0\,s) = (3.0\,m)(1.6 + 1.2t)\,s^{-1}\Big|_{t=2.0\,s} = (3.0\,m)(4.0\,s^{-1}) = 12.0\,m/s$

$a_{tan}(2.0\,s) = R\alpha(2.0\,s) = (3.0\,m)(1.2\ \text{rad}/s^2) = 3.6\,m/s^2$

$a_R(2.0\,s) = (\omega(2.0\,s))^2 R = \left[(1.6 + 1.2t)\,s^{-1}\Big|_{t=2.0\,s}\right]^2(3.0\,m) = (4.0\,s^{-1})^2(3.0\,m) = 48\,m/s^2$

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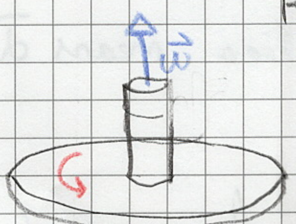
Vector nature of angular quantities

We can define pseudovectors $\vec{\omega}$ and $\vec{\alpha}$ so that we can use the power of vector algebra in our calculations involving rotation



Right-hand rule convention

when the fingers of the right hand are curled around the rotation axis and point in the direction of rotation, the thumb points in $\vec{\omega}$.

Note: nothing in the dish is moving in direction $\vec{\omega}$!

Note: if the axis of rotation is fixed then $\vec{\omega}$ and $\vec{\alpha}$ are on the same line, (but maybe with opposite directions)

$\vec{\omega}$ and $\vec{\alpha}$ are pseudovectors since they don't behave like vectors under reflection (like any cross product). This will become important later, but not of any concern at this level

Constant angular acceleration

The angular counterparts of constant linear acceleration equations of motion are derived similarly

Angular	Linear	
$\omega = \omega_0 + \alpha t$	$v = v_0 + at$	constant a, α
$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$	$x = v_0 t + \frac{1}{2}at^2$	constant a, α
$\omega^2 = \omega_0^2 + 2\alpha\theta$	$v^2 = v_0^2 + 2ax$	constant a, α
$\bar{\omega} = \frac{\omega + \omega_0}{2}$	$\bar{v} = \frac{v + v_0}{2}$	constant a, α

Note: constant angular acceleration means $\bar{\alpha} = \alpha$

ex: Centrifuge acceleration

A centrifuge rotor is accelerated from rest to 20,000 rpm in 30s

a) what is the average angular acceleration?

b) Through how many revolutions has the centrifuge rotor turned during its acceleration period, assuming constant acceleration

a) $\omega_f = 2\pi f = (2\pi \text{ rad/rev}) \frac{(20000 \text{ rev/min})}{(60 \text{ s/min})} = 2100 \text{ rad/s}$

$\omega_0 = 0$

$\bar{\alpha} = \frac{\Delta\omega}{\Delta t} = \frac{\omega_f - \omega_0}{\Delta t} = \frac{2100 \text{ rad/s} - 0}{30\text{s}} = 70 \text{ rad/s}^2$

b) $\theta = 0 + \frac{1}{2}(70 \text{ rad/s}^2)(30\text{s})^2 = 3.15 \times 10^4 \text{ rad}$

$\frac{3.15 \times 10^4 \text{ rad}}{2\pi \text{ rad/rev}} = 5.0 \times 10^3 \text{ rev}$

or

$\theta = \frac{\omega_f^2 - \omega_0^2}{2\alpha} = \frac{(2100 \text{ rad/s})^2 - 0}{2(70 \text{ rad/s}^2)} = 3.15 \times 10^4 \text{ rad}$

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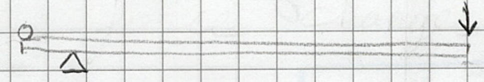
Torque

Now we move to rotational dynamics

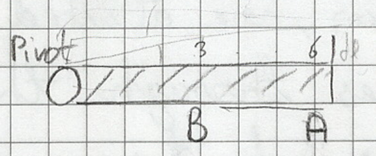
Give me a place to stand and with a lever
I shall move the world - Archimedes



The effect of a force on a rigid body depends on where and in which direction it is applied to.

The perpendicular distance from the axis of rotation to the line which the force acts is called the "lever arm" or "moment arm"

The tangential acceleration due to a contact force on a body results in greater angular acceleration as it is applied more distant from the axis of rotation



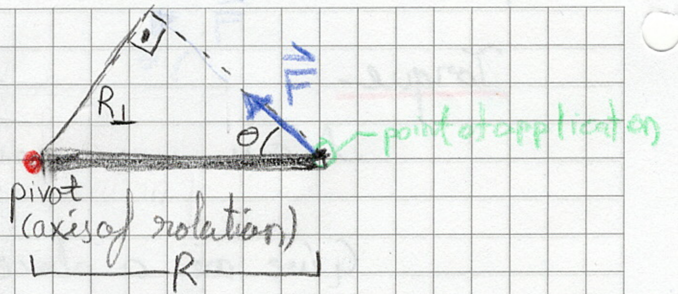
the same force applied in A will create more angular acceleration

In fact, the angular acceleration is proportional to the product of lever arm and force

Let's define angular counterpart of force

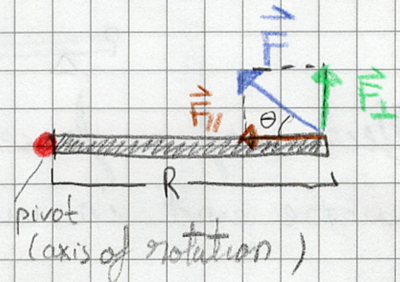
Torque $\tau \propto \alpha$

$$\boxed{\tau = R_\perp F}$$



or, equivalently

$$\boxed{\tau = R F_\perp}$$



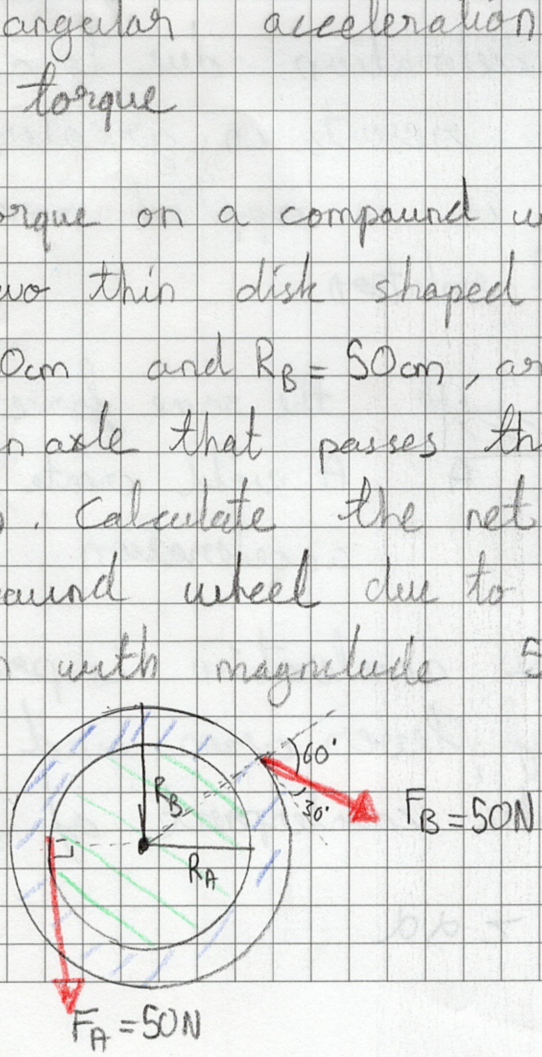
these are equal because $\tau = RF\sin\theta$

Torque has units m.N (and it is not work)

The angular acceleration is proportional to net torque

ex: Torque on a compound wheel

Two thin disk shaped wheels, of radii $R_A = 30cm$ and $R_B = 50cm$, are attached to each other on an axle that passes through the center of each. Calculate the net torque on this compound wheel due to indicated forces each with magnitude 50N

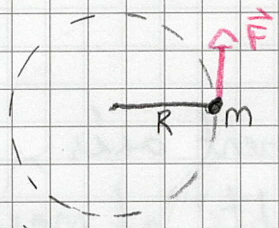


$$\tau = R_A F_A - R_B F_B \sin 60°$$

$$= (0.30m)(50N) - (0.50m)(50N) \times (0.866)$$

$$= -6.7\, m\,N$$

Rotational dynamics



Consider a particle rotating in a circle of radius R. Single force $\vec{F}$ acts tangentially

$$a_{tan} = R\alpha \ ; \ \tau = RF$$

$$F = ma \rightarrow F = mR\alpha \rightarrow \boxed{RF} = RmR\alpha$$

$$\boxed{\tau = mR^2\alpha}$$ single particle

This equation is in the same spirit but $m \rightarrow mR^2$. mR^2 represents here the rotational inertia of the particle and it is called the "moment of inertia"

Now consider rotating a rigid object. Use the above equation for all the constituents that form the said object

$$\boxed{\sum \tau_i = \left(\sum m_i R_i^2\right)\alpha}$$ fixed axis

(since $\alpha_i = \alpha$ for all particles in a rigid body)

moment of inertia then is

$$\boxed{I = \sum m_i R_i^2 = m_1 R_1^2 + m_2 R_2^2 + \cdots}$$ discrete case moment of inertia

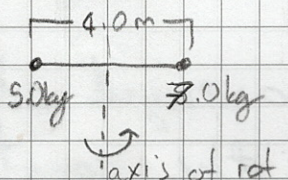
then

$$\boxed{\sum \tau = I\alpha}$$ axis fixed in inertial frame

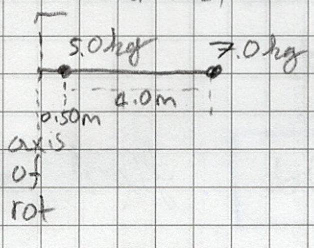
$$\boxed{\left(\sum \tau\right)_{CM} = I_{CM}\alpha_{CM}}$$ axis fixed in direction accelerating frame

i.e. you'll need to calculate everything using CM if the object also does a translational motion (i.e. a wheel)

Example: Two weights on a bar: different axis, different I. Two small "weights" of mass 5.0 kg and 7.0 kg are mounted 4.0 m apart on a light rod (ignorable mass) as shown. Calculate moment of inertia when system is rotated a) axis halfway between b) rotated 0.50m to the left of 5.0 kg mass

a) 

$$I = \sum mR^2 = (5.0\,kg)(2.0\,m)^2 + (7.0\,kg)(2.0\,m)^2 = 48\,kg.m^2$$

b) 

$$I = \sum mR^2 = (5.0\,kg)(0.5\,m)^2 + (7.0\,kg)(4.5\,m)^2$$
$$= 1.3\,kg\,m^2 + 142\,kg\,m^2$$
$$= 143\,kg.m^2$$

- the moment of inertia is vastly different when different axes of rotation is chosen
- masses closer to the axis have much less contribution to moment of inertia

For most ordinary objects, an integration needs to be performed for calculating I

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when calculating is difficult, the I can be measured experimentally

Rotational motion

1.) As always draw a clear diagram

2.) Choose the objects that will be the system

3.) Draw a free-body diagram for the object under consideration (for each individual object)

- All forces
- Only on that object
- where they act (to calculate torque)
- Gravity acts at CG

4) Identify axis of rotation, and calculate torques about it. Use right hand rule to assign + and −

5) Apply Newton's second law for rotation

$$\underset{m \cdot N}{\Sigma \tau} = \underset{kg \cdot m^2}{I} \underset{rad/s^2}{\alpha}$$

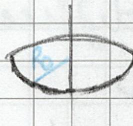
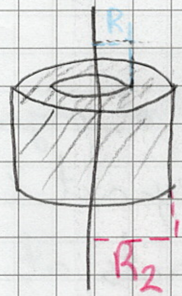
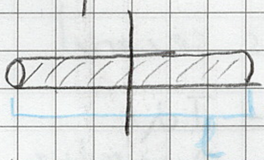
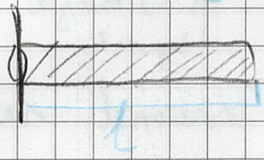
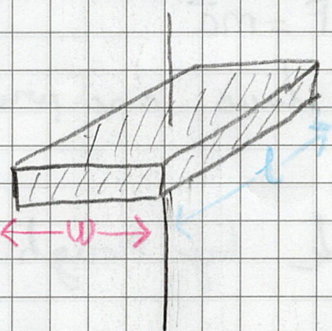
6) Apply Newton's second law for translation

$$\Sigma \vec{F} = m\vec{a}$$

(and other laws and principles as needed)

7) Solve

8) Error check by rough estimate

Object	Axis		Moment of Inertia
Thin hoop radius R_0	Through center		MR_0^2
Thin hoop radius R_0 width w	Through central diameter		$\frac{1}{2}MR_0^2+\frac{1}{2}Mw^2$
Solid cylinder radius R_0	Through center		$\frac{1}{2}MR_0^2$
Hollow cylinder inner R_1 outer R_2	Through center		$\frac{1}{2}M(R_1^2+R_2^2)$
Uniform sphere radius r_0	Through center		$\frac{2}{5}Mr_0^2$
Long uniform rod length l	Through center		$\frac{1}{12}Ml^2$
//	Through end		$\frac{1}{3}Ml^2$
Rectangular thin plate length l width w	Through center		$\frac{1}{12}M(l^2+w^2)$

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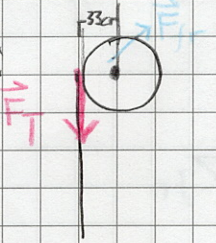
Ex: A heavy pulley: A 15.0-N force (represented by $\vec{F}_T$) is applied to a cord wrapped around a pulley of mass $M = 4.00\,kg$ and radius $R_0 = 33\,cm$. The pulley accelerates from rest to an angular speed of $30.0\,rad/s$ in $3.00\,s$. If there is a frictional torque of $\tau_{fr} = 1.10\,m.N$ at the axle, determine the moment of inertia of the pulley (rotating about its center)

1: 

2: the system is the pulley

3: ignore gravity and hemelle contact force (no contribution)

4) $\tau_T = R_0 F_T$ $\quad \tau_{fr} = 1.10\,m.N$

5) $\Sigma\tau = (0.330m)(15.0N) - 1.10\,m\,N$
$= 3.85\,m.N$

$$\alpha = \bar{\alpha} = \frac{\Delta\omega}{\Delta t} = \frac{30.0\,rad/s - 0}{3.00\,s} = 10\,rad/s^2$$

6) None needed

7) $\Sigma\tau = I\alpha \rightarrow I = \frac{\Sigma\tau}{\alpha} = \frac{3.85\,mN}{10\,rad/s^2} = 0.385\,kg\,m^2$

8) error check:

$$I \tilde{=} \frac{1}{2}MR_0^2 = \frac{1}{2}(4.0\,kg)(0.330\,m)^2 = 0.218\,kg\,m^2$$

(probably this pulley has more mass toward the edges, ie its closer to thin hoop $I = MR_0^2 = 0.436\,kg\,m^2$)

A heavy pulley and a bucket

Consider the same pulley (with the same friction), with a bucket of mass 1.53 kg attached. Cord has negligible mass, does not slip or stretch

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a) Calculate the angular acceleration of the pulley and the linear acceleration of the bucket

b) Determine the angular velocity ω of the pulley and the linear velocity v of the bucket at $t = 3.00\,s$ after the motion starts

→ Notice the tension on the string must be less than the weight of the bucket since the bucket accelerates. Thus we need to calculate tension. Divide the question into two parts

Pulley :

$I\alpha = \Sigma \tau = R_0 F_T - \tau_{fr}$

bucket :

$mg - F_T = ma$

no slip/stretch → pulley contact point with the cord has same linear acceleration as the cord

$$a_{tan} = a = R_0 \alpha$$

$$I\alpha = \Sigma \tau = R_0 F_T - \tau_{fr} = R_0 (mg - mR_0 \alpha) - \tau_{fr} = mgR_0 - mR_0^2 \alpha - \tau_{fr}$$

$$\rightarrow \alpha = \frac{mgR_0 - \tau_{fr}}{I + mR_0^2} = \frac{(15.0\,N)(0.330\,m) - 1.10\,m\,N}{0.385\,kg\,m^2 + (1.53\,kg)(0.330\,m)^2}$$

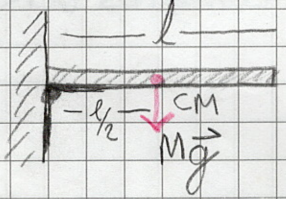
$$= 6.98\ rad/s^2$$

$$a = R_0 \alpha = (0.330\,m)(6.98\,rad/s^2) = 2.30\ m/s^2$$

b) Since α is constant, after $3.00\,s$ $\omega = \omega_0 + \alpha t = 0 + (6.98\,rad/s^2)(3.00\,s)$
$= 20.9\,rad/s$

$v = R_0 \omega = (0.330\,m)(20.9\,rad/s) = 6.91\,m/s$

Ex Rotating rod. A uniform rod of mass M and length l can pivot freely (i.e. ignore friction) about a hinge or pin attached to the case of a large machine. The rod is held horizontally, then released. At the moment of the release (when you are no longer exerting a force holding it up), determine a) the angular acceleration of the rod b) the linear acceleration of the tip of the rod



$$\alpha = \frac{\tau}{I} = \frac{Mg\, l/2}{\frac{1}{3}Ml^2} = \frac{3}{2}\frac{g}{l}$$

(as the rod descents $|\vec{F}_g|$ is the same, but the τ_g is not, so α is not constant

b) $a_{tan} = l\alpha = \frac{3}{2}g$

this is $>g$ thus an object at the edge will be left behind

on the other hand at CM $a_{tan} < g$ $(\frac{3}{4}g)$

Determining moments of inertia

By experiment : $I = \Sigma\tau/\alpha$

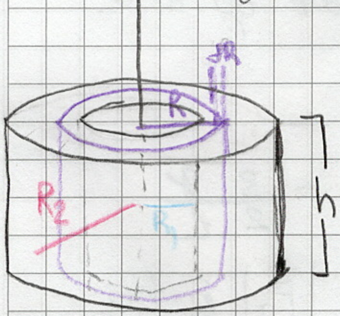
Using calculus (only viable for simple systems)

$$\boxed{I = \int R^2 dm}$$

Ex Cylinder, solid or hollow

a) Show that the moment of inertia of a uniform hollow cylinder of inner radius R_1 outer radius R_2, and mass M, is $I = \frac{1}{2}M(R_1^2+R_2^2)$ if the rotation axis is through the center along the axis of symmetry

b) Obtain the moment of inertia of a solid cylinder



uniform $\rightarrow dm = \rho\, dV$

for a thin disk at R with thickness dR

$$V = \underbrace{2\pi R dR}_{\text{base area}}\ \underbrace{h}_{\text{height}}$$

$$dm = 2\pi \rho h R dR$$

$$I = \int R^2 dm = \int_{R_1}^{R_2} 2\pi\rho h R^3 dR = 2\pi\rho h \left[\frac{R_2^4 - R_1^4}{4}\right]$$

$$= \frac{\pi\rho h}{2}(R_2^4 - R_1^4) = \frac{\pi\rho h}{2}(R_2^2-R_1^2)(R_2^2+R_1^2)$$

Let's calculate ρ

$$V = (\pi R_2^2 - \pi R_1^2) h$$

$$M = \rho V = \rho\pi(R_2^2 - R_1^2) h$$

$$I = \frac{1}{2}M(R_1^2+R_2^2)$$

b) for a solid cylinder $R_1 = 0 \quad R_2 = R_0$

$$I = \frac{1}{2}MR_0^2$$

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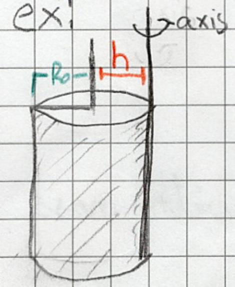
The parallel axis theorem

This theorem relates the moment of inertia I of an object of total mass M about any axis, to its moment of inertia through CM (I_{CM}) and parallel to said axis.

If the two axes are a distance h apart

$$\boxed{I = I_{CM} + Mh^2}$$ (Parallel axis theorem)

ex:



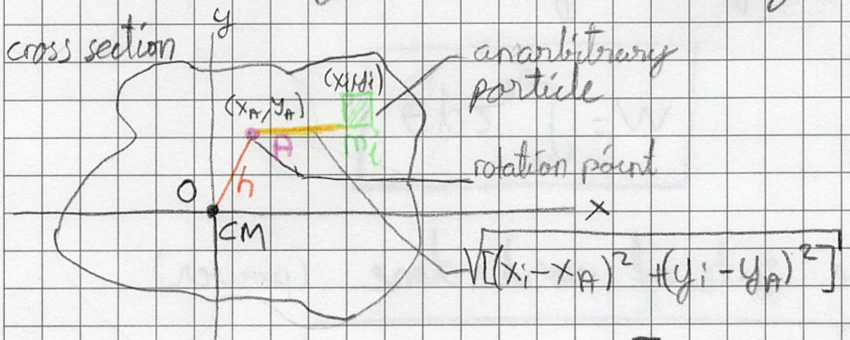
Determine the moment of inertia of a solid cylinder of radius R_0 and mass M about an axis tangent to its edge and parallel to its symmetry axis

$$h = R_0$$

$$I = \underbrace{I_{CM}}_{\frac{1}{2}MR_0^2} + M\overset{R_0^2}{h^2} = \frac{3}{2}MR_0^2$$

proof of the parallel axis theorem

Choose the coordinate system such that CM is at the origin, and I_{CM} is along z axis



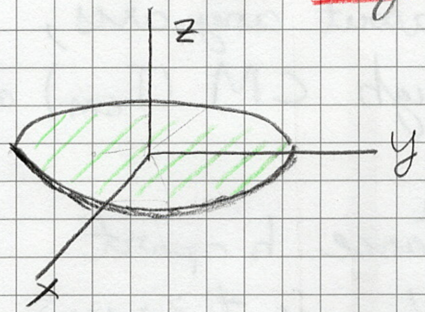
The object is rotated about an axis passing through A

$$I = \sum m_i[(x_i-x_A)^2+(y_i-y_A)^2] = \underbrace{\boxed{\sum m_i(x_i^2+y_i^2)}}_{I_{cm}} - 2x_A\underbrace{\boxed{\sum m_i x_i}}_{=0} - 2y_A\underbrace{\boxed{\sum m_i y_i}}_{=0} + \underbrace{\boxed{(\sum m_i)}}_{M}\underbrace{\boxed{(x_A^2+y_A^2)}}_{h^2}$$

CM is at O

The perpendicular axis theorem

This can only be applied to flat objects or objects of uniform thickness whose thickness can be neglected



$$\boxed{I_z = I_x + I_y}$$

proof: $I_x = \sum m_i y_i^2 \quad I_y = \sum m_i x_i^2 \quad I_z = \sum m_i (x_i^2 + y_i^2)$

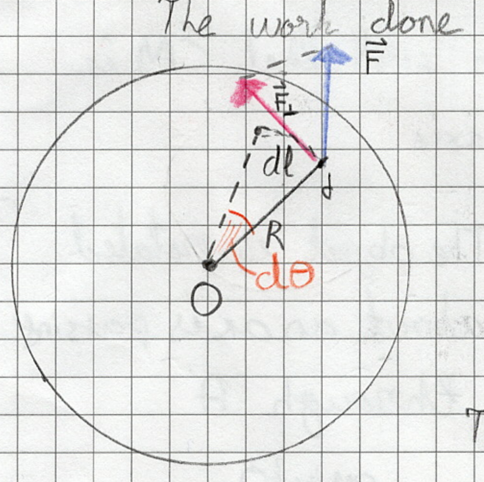
Rotational Kinetic Energy

For any rotating object, any small section with mass m_i has a linear speed $v_i = R_i w$. Thus

$$K = \sum \left(\frac{1}{2} m_i v_i^2\right) = \sum \left(\frac{1}{2} m_i R_i^2 w^2\right) = \frac{1}{2} \underbrace{\left(\sum m_i R_i^2\right)}_{I} w^2$$

$$\boxed{K = \frac{1}{2} I w^2}$$

The work done:



$$W = \int \vec{F} \cdot d\vec{l} = \int \overbrace{F_\perp R}^{\tau} d\theta$$

$$\boxed{W = \int_{\theta_1}^{\theta_2} \tau \, d\theta}$$

The rate of work done (power)

$$\boxed{P = \frac{dW}{dt} = \tau \frac{d\theta}{dt} = \tau w}$$

The work-energy principle holds for rotating bodies

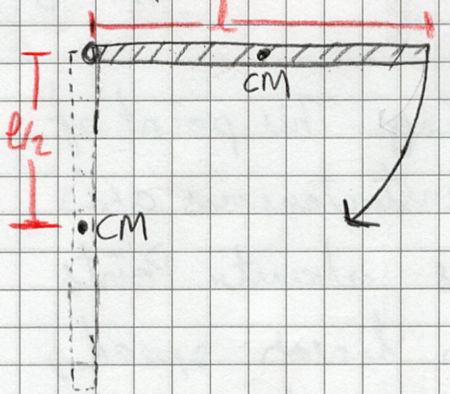
$$\tau = I\alpha = I\frac{d\omega}{dt} = I\frac{d\omega}{d\theta}\overbrace{\frac{d\theta}{dt}}^{\omega} = I\omega\frac{d\omega}{d\theta}$$

$$\tau d\theta = I\omega d\omega$$

$$\boxed{W = \int_{\theta_1}^{\theta_2} \tau d\theta = \int_{\omega_1}^{\omega_2} I\omega d\omega = \frac{1}{2}I\omega_2^2 - \frac{1}{2}I\omega_1^2}$$

The work done in rotating an object from θ_1 to θ_2 is equal to the change in rotational kinetic energy of the system.

Ex: Rotating rod. A rod of mass M is pivoted on a frictionless hinge at one end. The rod is held at rest horizontally and then released. Determine the angular velocity of the rod when it reaches the vertical position, and the speed of the rod's tip at this moment



Using Work-energy principle

$$W = Mg\,l/2$$

$$\frac{1}{2}I\omega^2 - 0 = Mg\,l/2$$

$$\omega = \sqrt{\frac{3g}{l}}$$

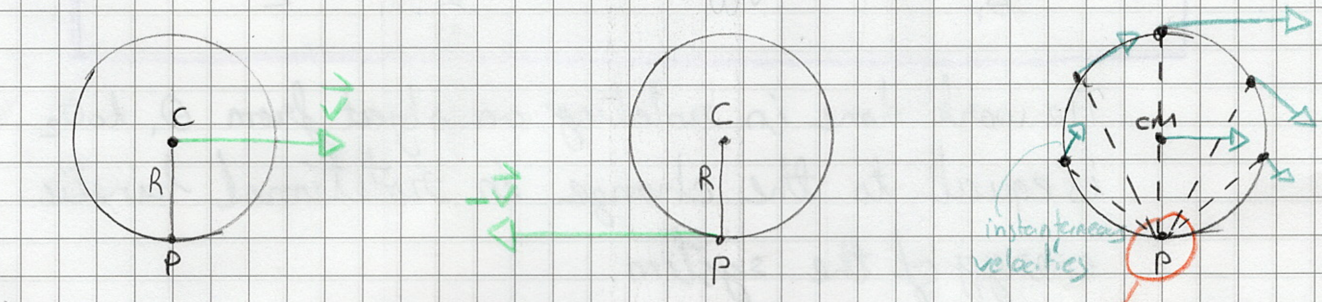
$$v = l\omega = \sqrt{3gl}$$

(note: an object that falls vertically has $v = \sqrt{2gl}$)

Rotational + Translational motion; Rolling

Rolling Without slipping

- Depends on static friction
- Rolling object's point of contact with the ground is at rest at each moment.
- Involves both rotation and translation



Earth reference frame

wheel's reference frame

P point is the instantaneous axis (perpendicular to the page)

using the wheel reference frame, we can relate translational speed of the wheel with its angular speed as

$$\boxed{v = R\omega} \quad \text{(rolling without slipping)}$$

When the wheel rolls without slipping. This point of contact can be considered as an instantaneous axis of rotation which the wheel rotates about. Points closer to the ground has smaller linear speed, whereas the points farther away has greater speed.

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Total kinetic energy

The total kinetic energy is the rotational kinetic about the instantaneous axis at instantaneous point of contact, since the object undergoes a pure rotation around the instantaneous axis

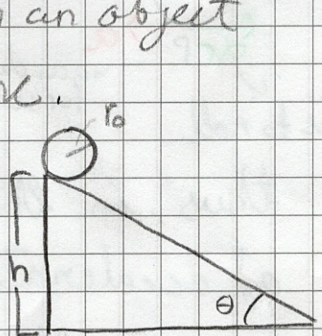
$$K_{tot} = \frac{1}{2} I_p \omega^2$$

$$I_p = I_{cm} + MR^2 \quad \text{(parallel axis theorem}$$

$$K_{tot} = \frac{1}{2} I_{cm} \omega^2 + \frac{1}{2} M \underbrace{R^2 \omega^2}_{v^2}$$

$$\boxed{K_{tot} = \frac{1}{2} I_{cm} \omega^2 + \frac{1}{2} M v^2}$$

Ex: sphere rolling down an incline. What will be the speed of a solid sphere of mass M and radius r_0 when it reaches the bottom of an incline if it starts from rest at a vertical height H and rolls without slipping? (Assume no slip is due to static friction which does no work)
Compare your result to that for an object sliding down a frictionless incline.



$$E = \frac{1}{2} M v^2 + \frac{1}{2} I_{cm} \omega^2 + Mgy$$

$$0 + 0 + MgH = \frac{1}{2} M v^2 + \frac{1}{2} \underbrace{I_{cm}}_{\frac{2}{5} M r_0^2} \underbrace{\omega^2}_{\left(\frac{v}{r_0}\right)^2} + 0$$

$$\left(\frac{1}{2} + \frac{1}{5}\right) v^2 = gH \rightarrow v = \sqrt{\frac{10}{7} gH} < \sqrt{2gH}$$

(sliding object)

The sphere is slower than the sliding box not because of friction thermalisation, but since some gravitational potential energy got converted to rotational kinetic energy. Static friction does no work ($d=0$)

Since translational movement can be investigated seperately, we can apply

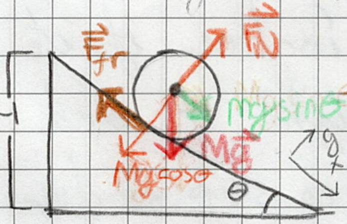
$$\sum \tau_{CM} = I_{CM}\,\alpha_{CM}$$

WARNING! You shall only attempt this if

1) The axis is fixed in an inertial frame
2) The axis is fixed in direction and passes through CM

Otherwise it becomes too unwieldy

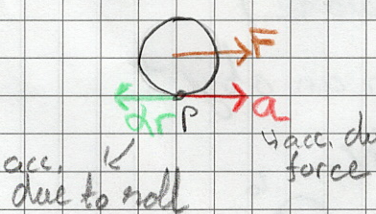
Ex: Analysis of a sphere on an incline using forces



$I_0 = \frac{2}{5}Mr^2 \qquad I = I_0 + Mr^2$

$I = \frac{7}{5}Mr^2 \qquad \alpha = \frac{\tau}{I} = \frac{Mg\sin\theta\, r}{\frac{7}{5}Mr^2} = \frac{5g\sin\theta}{7r}$

$a = \alpha r = \frac{5g\sin\theta}{7} < \frac{F}{m} = g\sin\theta$



Since the object is rolling without slipping, point P must be stationery thus, in this case F_{fr} must be in the direction of acceleration due to rolling

$Mg\sin\theta - F_{fr} = Ma$

$F_N - Mg\cos\theta = 0 \rightarrow F_N = Mg\cos\theta$

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$$F_{fr}\, r_0 = \left(\frac{2}{5}Mr_0^2\right)\alpha = \frac{2}{5}M(r_0^2\alpha)$$

$$F_{fr} = \frac{2}{5}Ma$$

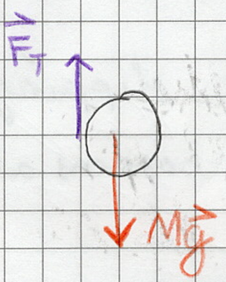
$$Mg\sin\theta - \frac{2}{5}Ma = Ma$$

$$a = \frac{5}{7}g\sin\theta$$] exactly as above

$$F_{fr} = \frac{2}{7}Mg\sin\theta$$ → the point p stays at rest

Ex: falling yo-yo

String is wrapped around a uniform solid cylinder of mass M and radius R, and the cylinder starts falling from rest. As the cylinder falls find its acceleration and tension in the string



$$Ma = \Sigma F = Mg - F_T$$

$$\Sigma \tau_{cm} = I_{cm}\alpha_{cm}$$

$$F_T R = \frac{1}{2}MR^2\alpha = \frac{1}{2}MR^2\left(\frac{a}{R}\right) = \frac{1}{2}MRa$$

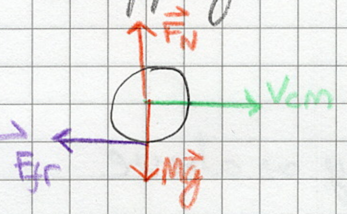
$$F_T = \frac{1}{2}Ma$$

$$Ma = Mg - \frac{1}{2}Ma \rightarrow a = \frac{2}{3}g$$

$$F_T = \frac{1}{3}Mg$$

Ex: Rolling ball slips

A bowling ball of mass M and radius r_0 is thrown along a level surface so that initially ($t=0$) it slides with a linear speed v_0 but does not rotate. As it slides, it begins to spin, and eventually rolls without slipping. How long does it take to begin rolling without slipping?



$$Ma_x = \Sigma F_x = -F_{fr} = -\mu_k F_N = -\mu_k Mg$$

(μ_k because the ball is sliding)

$$a_x = -\mu_k g$$

$$V_{CM} = V_0 + a_x t = V_0 - \mu_k g t$$

$$I_{CM}\alpha_{CM} = \Sigma \tau_{CM} \rightarrow \frac{2}{5} M r_0^2 \alpha_{CM} = F_{fr} r_0 = \mu_k M g r_0$$

$$\alpha_{CM} = 5\mu_k g / 2r_0 \quad \text{(constant)}$$

$$w_{CM} = w_0 + \alpha_{CM} t = 0 + \frac{5\mu_k g t}{2 r_0}$$

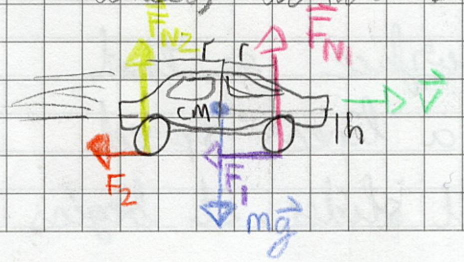
The condition for rolling without slipping is $V_{cm} = w_{cm} r_0$. Let's say this happens at $t = t_1$

$$V_0 - \mu_k g t_1 = \frac{5\mu_k g t_1 r_0}{2 r_0}$$

$$t_1 = \frac{2V_0}{7\mu_k g}$$

Ex: Braking car:

Why do we need bigger brake pads on the front wheels? (Assume wheels do not slip)



$$F_1 = \mu_s F_{N1} \qquad F_2 = \mu_s F_{N2}$$

$$F_1 + F_2 = ma$$

$$r F_{N_1} - r F_{N_2} - h F_1 - h F_2 = 0$$

(so that the car does not flip)

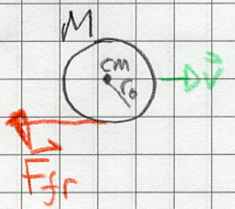
$$r(F_{N_1} - F_{N2}) - h\,\mu(F_{N_1} + F_{N2}) = 0$$

$$F_{N_1}(r - h\mu) - F_{N_2}(r + h\mu) = 0$$

$$F_{N_1} = F_{N_2}\frac{(r+h\mu)}{(r-h\mu)}$$

so F_{N_1} is greater than F_{N2}, meaning more force on front tires. If the brake pads were equal, front ones would've worn out much faster. (note μ is not μ_s, since the car is not about to skid)

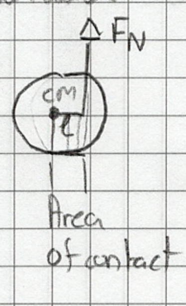
Why does a rolling sphere slow down?



A sphere of mass M and radius r_0 rolling on a horizontal flat surface comes to a stop.

The force that is decelerating the sphere acts to increase the angular acceleration!

Actually we have a shortcoming in our model. The "point" of contact in actuality is "area of contact" in a more realistic material



Since the impulse on the front of the wheel is larger, there is a larger force towards the front of the are of contact. We can model this effect as a net normal force with a moment arm ℓ from CM. Hence this force stops the rolling this is why you need higher pressures on road bikes (less "friction") whereas lower pressure on mountain bikes.