Newton's Law of Universal Gravitation

Sir Isaac Newton on gravity:

- falling objects accelerate → there must be a force
- the force is exerted by something
- No matter where on earth, there is gravity and it is towards the center of the earth

→ Earth is exerting force on objects at its surface

This was quite controversial at the time
"force at a distance" (without contact)

→ Astronomy (What better suits as a laboratory than the sky for this?)

magnitude of the gravitational motion

at surface of earth ; $a = 9.80\ m/s^2 = g$

a_R of moon $\cong \frac{1}{3600} g$

$$\frac{\overset{\nearrow 6380\,km}{r \text{ @ Earth surface}}}{\underset{\searrow 384.000}{\text{distance of moon}}} = \frac{1}{60}$$

$F \propto \frac{1}{r^2}$

this force also depends on the masses

—must depend on both masses (Newton's third law)

$$F \propto \frac{m_E m_B}{r^2}$$

investigating the solar system further, Newton proposed

Every particle in the universe attracts every other particle with a force that is proportional to the product of their masses and inversely proportional to the square of the distances between them.

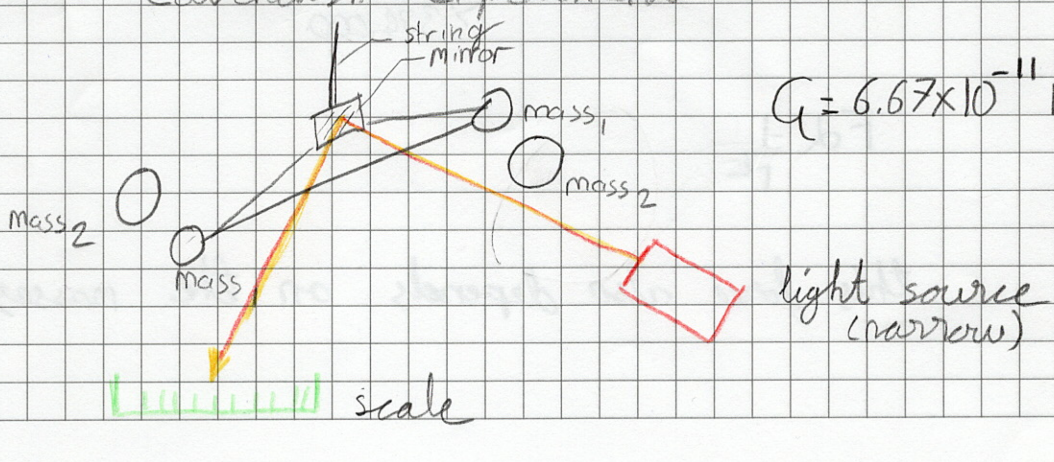
$$\boxed{F = G\frac{m_1 m_2}{r^2}}$$ Newton's law of universal gravitation

m_1, m_2 : masses of the particles

r : distance between the particles

The value of G must be very small since we don't feel the attraction between ordinary every-day objects

Cavendish experiment



$G = 6.67 \times 10^{-11}\ Nm^2/kg^2$

What is your (gravitational) attraction?

70 kg vs 50 kg $\frac{1}{2}$ m (btw centers) (contact d.)

$$F = \frac{(6.67\times10^{-11}\ N\,m^2/kg^2)(50kg)(70kg)}{(0.5m)^2} \simeq 10^{-6}N$$

Spacecraft at $2r_E$

What is the force of gravity acting on a 2000 kg spacecraft when it orbits two earth radii from the Earth's center? $m_E = 5.98\times10^{24}$ kg

$r_E = 6380$ kg

$m_s = 2000$ kg

$F_{surface} = mg \qquad F_{r=2r_E} = \frac{1}{4}mg = \frac{1}{4}(2000kg)(9.80\ m/s^2) = 4900N$

Force on the Moon

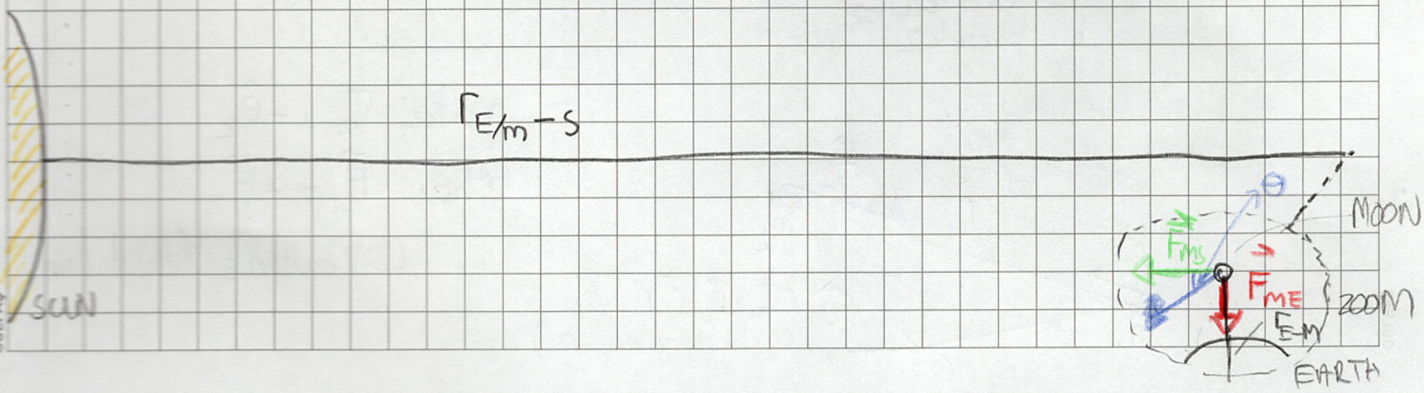
Find the net force on the moon due to earth and the sun at right angle

$m_M = 7.35\times10^{22}$ kg $\qquad m_E = 5.98\times10^{24}$ kg $\qquad m_S = 1.99\times10^{30}$ kg

$r_{E-M} = 3.84\times10^{8}$ m $\qquad r_{E/M-S} = 1.50\times10^{11}$ m

r_{E-M} is $\frac{1}{1000}$ th of $r_{E/M-S}$

m_S is %99.8 — %99.9 of the entire solar system



$$F_{MS} = \frac{(6.67\times10^{-11}\,Nm^2/kg^2)(7.35\times10^{22}\,kg)(1.99\times10^{30}\,kg)}{(1.50\times10^{11}\,m)^2}$$

$$= 4.34\times10^{20}\,N$$

$$F_{ME} = \frac{(6.67\times10^{-11}\,Nm^2/kg^2)(7.35\times10^{22}\,kg)(5.98\times10^{24}\,kg)}{(3.84\times10^{8}\,m)^2}$$

$$= 1.99\times10^{20}\,N$$

at right angle

$$F = \sqrt{(1.99\times10^{20}N)^2 + (4.34\times10^{20}N)^2} = 4.77\times10^{20}\,N$$

$$\theta = \tan^{-1}(1.99/4.34) = 24.6^\circ$$

Q: why moon does not get stollen by sun?

Q: Why the moon has greater effect on tides?

(self research topic)

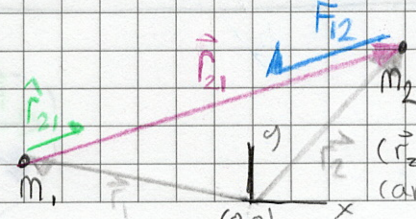
Vector Form of Newton's Law of Universal Gravitation

$$\vec{F}_{12} = -G\frac{m_1 m_2}{r_{21}^2}\hat{r}_{21}$$

unit vector along the line connecting two particles

$$\hat{r}_{21} = \vec{r}_{21}/r_{21}$$

vector force on particle 1 by particle 2



$(\vec{r}_{21} = \vec{r}_2 - \vec{r}_1)$

(arbitrary origin)

notice $\vec{r}_{21} = -\vec{r}_{12}$

thus $\vec{F}_{21} = -\vec{F}_{12}$

(Newton's third law)

when many particles interact with each other

$$\vec{F}_1 = \vec{F}_{12} + \vec{F}_{13} + \vec{F}_{14} \cdots \vec{F}_{1N} = \sum_{i=2}^{n} \vec{F}_{1i}$$

→ number of particles

↳ no self interaction

The three body problem (not the book)

... is the problem of taking the initial positions and velocities of three point masses and solving their subsequent equations of motion using Newton's laws of motion and Newton's law of universal gravitation

There is no general analytical solution to the three-body problem given by simple algebraic expressions and integrals

→ Special case solutions

Gravity Near the Earth's Surface; Geophysical applications

$$mg = \frac{G\, m\, M_E}{r_E^2} \rightarrow g = \frac{G\, M_E}{r_E^2}$$

mass of earth

weight

radius of earth

once Cavendish obtained G, he was able to estimate M_E (~1798)

$$M_E = \frac{g r_E^2}{G} = \frac{(9.80\, m/s^2)(6.38 \times 10^6\, m)^2}{6.67 \times 10^{-11}\, N\, m^2/kg^2} = 5.98 \times 10^{24}\, kg$$

ex. Gravity on Everest. Estimate the effective value of g on the top of Mt. Everest (8850 mt)

$$g = G\frac{M_E}{r^2} = \frac{(6.67 \times 10^{-11}\, Nm^2/kg^2)(5.98 \times 10^{24}\, kg)}{(6.389}$$

$$= 9.77\, m/s^2$$ (0.3% less)

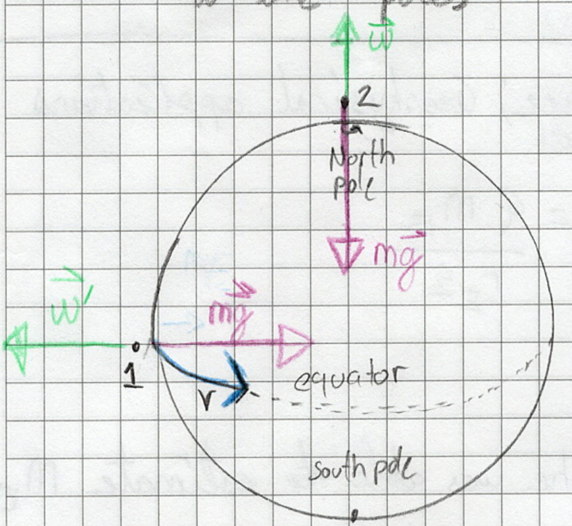
Note that we did not take into account the local density of Earth

→ variations due to local density is called gravity anomalies (on the order of $10^{-6} \sim 10^{-7}$)

gravimeters can measure 10^{-9} → survey oil & mineral fields

Effect of Earth's rotation on g:

Assuming the Earth is a perfect sphere, determine how the Earth's rotation affects the value of g at the equator compared to the poles



in situation 1, there is additional acceleration due to centripetal force

$$v = \frac{2\pi r_E}{1\text{day}} = \frac{(6.283)(6.38 \times 10^6 m)}{(8.64 \times 10^4 s)}$$

$$= 4.640 \times 10^2 m/s$$

$$m\vec{g} - \vec{w}' = \frac{mv^2}{r_E} \quad (1)$$

$$m\vec{g} - \vec{w} = 0 \quad (2)$$

$$w' = m\left(g - \frac{v^2}{r_E}\right) \qquad g' = \frac{w'}{m} = g - \frac{v^2}{r_E}$$

$$\Delta g = g - g' = \frac{v^2}{r_E} = \frac{(4.640 \times 10^2 m/s)}{(6.38 \times 10^6 m)} = 0.0337 \; m/s^2$$

$\simeq 0.3\%$

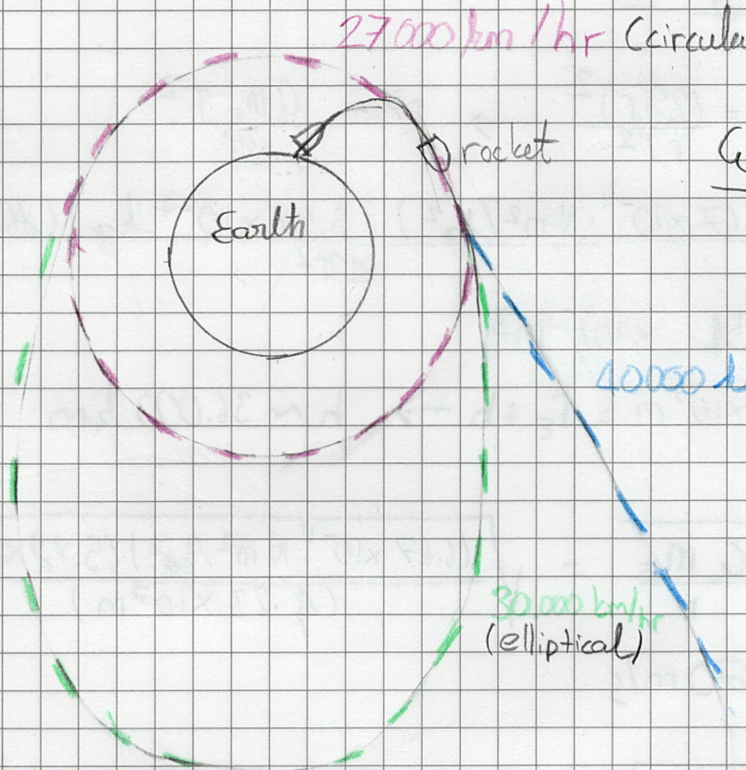
Chapter 6 P4 V1

Note: Earth is not a sphere (fatter by 21 km at equator)

Note: near the poles, centripetal force and gravitational force are not aligned, thus this becomes a 2D problem

Note: Earth as an inertial frame means we will have some error in our calculations (i.e %0.3 in Newton's second law in above case)

Sattellite motion



$$\frac{G\, m\, m_E}{(r)^2} = m\frac{v^2}{(r)}$$

$r = r_E + h$

what keeps a satellite up? its speed

a satellite is falling, but due to its tangential speed it keeps missing the ground

Ex: Geosynchronous satellite

Geosynchronous : stays above a fixed point on Earth only possible above equator.

(a) what must be height above ground

(b) speed?

(c) speed

to be geosynchronous, the satellite must have $T = 24\,hrs$

(a) $$G\frac{\cancel{m_{sat}} M_E}{r^2} = \cancel{m_{sat}}\frac{v^2}{r}$$

$$v = \frac{2\pi r}{T}$$

$$G\frac{m_E}{r^2} = \frac{(2\pi r)^2}{r T^2} \longrightarrow r^3 = \frac{G m_E T^2}{4\pi}$$

$$= \frac{(6.67\times10^{-11}\ Nm^2/kg^2)(5.98\times10^{24}\ kg)(86400\ s)^2}{4\pi^2}$$

$$= 7.54 \times 10^{22}\ m^3$$

$$r = 4.23\times10^7\ m = r_E + h \rightarrow h \sim 36.000\ km$$

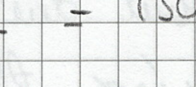
(b) $$v = \sqrt{\frac{G m_E}{r}} = \sqrt{\frac{(6.67\times10^{-11}\ Nm^2/kg^2)(5.98\times10^{24}\ kg)}{(4.23\times10^7 m)}}$$

$$= 3070\ m/s$$

(c) $$v \propto \sqrt{1/r} \qquad r = r_E + h = (6380 + 200)\ km$$

$$v' = v\sqrt{\frac{r}{r'}} = (3070\ m/s)\sqrt{\frac{42300\ km}{6580\ km}} = 7780\ m/s$$

Question : can you have a satellite that does not have earth at its exact center



if you want to catch a satellite in orbit as an astronaut, what do you do? Smaller orbit → faster
Slow down to rise

Weightlessness:

If you are accelerating towards earth at g then your apparent weight is

$w = mg - mg = 0$

(see the "vomit comet" of NASA which you can buy a ticket)

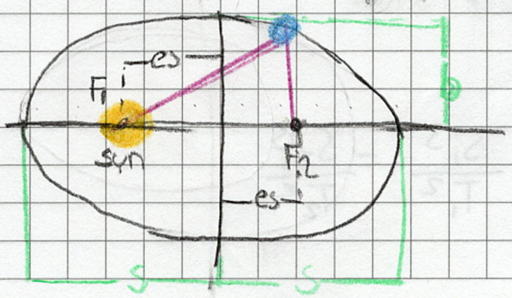
this is called "free fall"

this is the same thing an astronaut or a satellite "feels" when orbiting
true weightlessness should be far away from everything

Kepler's Laws and Newton's Synthesis

Johannes Kepler (1571-1630)
Tycho Brahe (1546-1601) (data)] German

More than half a century before Newton, Kepler wrote laws of planetary motion in our solar system



Kepler's first law: The path of each planet about the sun is an ellipse with the sun at one focus in the elipse:

$F_1P + F_2P$ = constant for all points

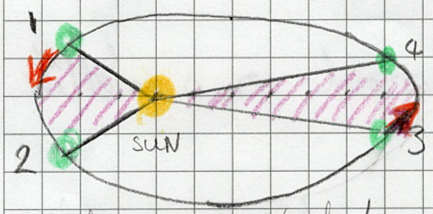
S: semimajor axis

eccentricity: e: ratio of the distance from either focus to the center divided by semimajor axis (thus, eS is the distance between center and focal point)

for a circle F_1 and F_2 are at the center, e=0

earth has an orbit with e=0.017, almost circular

b: semiminor axis



planets move fastest when they are closest to sun. Here the time it takes from 1 to 2 is equal to time it takes from 3 to 4

Kepler's second law:

Each planet moves such that an imaginary line drawn from sun to the planet sweeps out equal areas in equal periods of time

Kepler's Third Law:

The ratio of the squares of the periods of any two planets revolving about the Sun is equal to the ratio of the cubes of their semimajor axes

$$\left(\frac{T_1}{T_2}\right)^2 = \left(\frac{S_1}{S_2}\right)^3 \quad \text{or} \quad \frac{S_1^3}{T_1^2} = \frac{S_2^3}{T_2^2}$$

means S^3/T^2 should be same for every planet

Kepler → analysis of data → empirical results

Newton → Kepler's laws can be derived from Newton's laws of motion + universal gravitation

+ only $1/r^2$ dependence is possible for any reasonable gravitational law

Newton used Kepler's laws as evidence in favor of his law of universal gravitation

Let's derive Kepler's third law for a circular orbit using Newton's laws and gravitation

$$\Sigma F = ma$$

$$G\frac{m_1 M_S}{r_1^2} = m_1\frac{V_1^2}{r_1}$$

$$V_1 = \frac{2\pi r_1}{T_1}$$

$$G\frac{m_1 M_S}{r_1^2} = m_1\frac{4\pi^2 r_1}{T_1^2}$$

$$\frac{T_1^2}{r_1^3} = \frac{4\pi^2}{GM_S}$$

thus prooving $\left(\frac{T_1}{T_2}\right)^2 = \left(\frac{r_1}{r_2}\right)^3$

Warning: Kepler's third law only applies to objects orbiting the same attracting center. (i.e do not compare moon around earth to Mars around sun) ex: Where is Mars?

Kepler: Mars' period is 687 days (Earth days)

Determine mean distance of Mars from Sun

$\frac{687\,d}{365\,d/y} = 1.8\,yr$

$\frac{r_{ms}}{r_{ES}} = \left(\frac{T_m}{T_E}\right)^{2/3} = \left(\frac{1.8}{1}\right)^{2/3} = 1.52$

$1.52 \times 1.50 \times 10^{11}\,m = 2.28 \times 10^{11}\,m$

ex: The Sun's mass

Determine the mass of the sun given the $r_{ES} = 1.5 \times 10^{11}\,m$

soln.:

$T_E = 1yr = 3.16 \times 10^7\,s$

$$M_S = \frac{4\pi^2 r_{ES}^3}{G T_E^2} = 2.0 \times 10^{30}\,kg$$

Accurate measurements of orbitals → observations of "perturbations" (deviations)

Uranus orbital perturbations couldn't be attributed to known planets at the time → Neptune was found

pluto (1930s) due to perturbations of Neptune

1990s → planets revolving around other stars

Newton's laws are casual → Universe is a big machine working in a deterministic manner

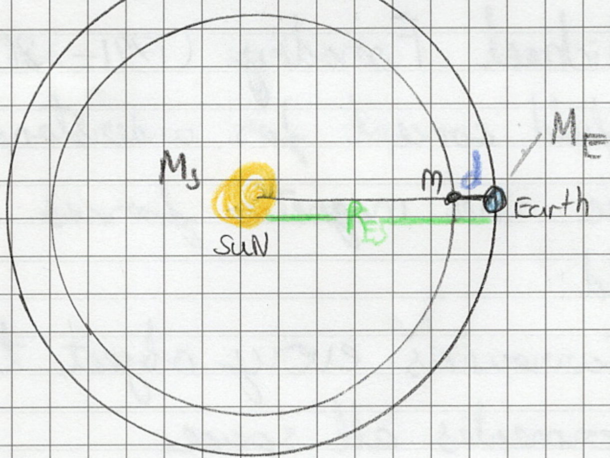
20th century: deterministic view is "modified" ex Lagrange point

The mathematician Joseph-Louis Lagrange

→ five special points in the vicinity of the Earth's orbit about the sun where a small satellite (mass m) can orbit the Sun with the same period T as earth

One of these lagrange points L1, lies between the Earth and the Sun

determine the distance of this point from Earth



Earth

$$\frac{G M_E M_S}{R_{ES}^2} = M_E \frac{v^2}{R_{ES}} = \frac{M_E}{R_{ES}} \frac{(2\pi R_{ES})^2}{T^2} \rightarrow \frac{G M_S}{R_{ES}^2} = \frac{4\pi^2 R_{ES}}{T^2}$$

satellite:

$$\frac{G M_S}{(R_{ES}-d)^2} - \frac{G M_E}{d^2} = \frac{4\pi^2 (R_{ES}-d)}{T^2}$$

$$\frac{G M_S}{R_{ES}^2}\left(1-\frac{d}{R_{ES}}\right)^{-2} - \frac{G M_E}{d^2} = \frac{4\pi^2 R_{ES}}{T^2}\left(1-\frac{d}{R_{ES}}\right)$$

Binomial expansion: $(1+x)^n \simeq 1+nx$ when $x \ll 1$

$x = d/R_{ES}$

$$\frac{GM_S}{R_{ES}^2}\left(1+\frac{2d}{R_{ES}}\right)-\frac{GM_E}{d^2}=\frac{4\pi^2 R_{ES}}{T^2}\left(1-\frac{d}{R_{ES}}\right)$$

$$\frac{4\pi^2 R_{ES}}{T^2} \rightarrow \frac{GM_S}{R_{ES}^2}$$

$$\frac{GM_S}{R_{ES}^2}\left(3\frac{d}{R_{ES}}\right)=\frac{GM_E}{d^2}$$

$$d=\left(\frac{M_E}{3M_S}\right)^{1/3}R_{ES}=1.0\times10^{-2}R_{ES}=1.5\times10^{6}\,km$$

small enough

L1: home of SOHO (Solar and Heliospheric observatory)

Gravitational field

Michael Faraday (1791-1867)
field concept for understanding
electric and magnetic forces

Gravitational field:
- Surrounds every object that has mass
- Permeates all space

A second object experience force due to field of first.

gravitational field: gravitational force per unit mass

$$\vec{g}=\vec{F}/m \quad (N/kg)$$

single mass:

$$\vec{g}=-\frac{GM}{r^2}\hat{r}$$

$\hat{r}$ points radially outward from M, (-) means it is attracting things. For multiple masses $\rightarrow$ vector sum of $\vec{g}$