Qubits, Operators and Measurement

A qubit is a quantum bit. It is similar to a classical bit in that it has two eigenstates - $|0\rangle$ and $|1\rangle$, but it is different from the classical bit since its expectation is a contiunious range of values

In general two level qubit systems are used, however there is no theoretical limitation for using multi level systems

qubit → two level system
qutrit → three level system
⋮
qudit → general name

A qubit system of 100 qubits can handle 2^{100} states (1,26765 E+30) while a qutrit system can handle 3^{100} states (5.15378 E+47)

100 qubit → 63 qutrits

Since it is more difficult to construct qutrits the mainstream QCs are currently based on qubits. Whether it is qubits, qutrits or any qudit each of these systems can run any algorithm that the others can.

Physical Qubit

A physical qubit is a two-level quantum mechanical system. It can be represented as a two dimensional complex Hilbert Space $\mathbb{C}^2$. The state of the qubit at any given time can be represented by a vector in this complex Hilbert space

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Qubit Operators (Gates)

In gate-based quantum computers, the operators which we use to evolve the state of the qubits are unitary, and therefore reversible. Some operators are involutive (they are their own inverses). A measurable quantity, or observable is Hermitian operator, thus the measurement in a QC outputs real values.

$$|0\rangle\langle 0| = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$|0\rangle\langle 1| = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$|1\rangle\langle 0| = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$|1\rangle\langle 1| = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

by suming these operators we can form an unitary matrix

$$|0\rangle\langle 1| + |1\rangle\langle 0| = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

This is actually the X or NOT operator

Representing Superposition of States

We represent a superposition of states as the linear combination of computational bases of the state space. Each term in the superposition has a complex coefficient or amplitude

$$|+\rangle := \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$|-\rangle := \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

These two states differ by a -1 in front of $|1\rangle$ state. This corresponds to a 180° phase difference by Euler's identity

$$e^{i\pi} = -1$$

Relative phases are of fundamental importance in QC algorithms! They allow constructive interference and destructive interference

$$\frac{1}{\sqrt{2}}\left(|+\rangle + |-\rangle\right) = \frac{1}{2}\left(|0\rangle + |1\rangle\right) + \frac{1}{2}\left(|0\rangle - |1\rangle\right) = |0\rangle$$

here $|1\rangle$ state interfere destructively, whereas $|0\rangle$ state interferes constructively

Another example

$$\frac{1}{\sqrt{2}}\left(|-\rangle - |+\rangle\right) = -|1\rangle$$

here $|0\rangle$ state interfere destructively and $|1\rangle$ state interfere constructively.

- $-$ in front of final state $|1\rangle$ effects entire state. This is called "global phase"

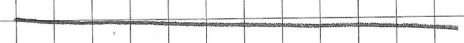
- $-|1\rangle$ is not $|1\rangle$ but global phase changes has no impact on quantum measurements ($|\langle\alpha|\alpha\rangle|^2$) hence $-|1\rangle$ and $|1\rangle$ state can be considered identical for practical purposes.
  - These two states are "equal up to a global phase"

Quantum Circuit diagrams

Circuit diagrams are used to depict quantum circuits. They resemble staff of music which is read in the same dir

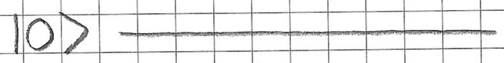
Barenco et al set forth the foundational operators. Fredkin and Toffoli added to this set two ternary operators

Start with a circuit line

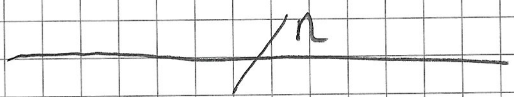


A line with no operator implies that the qubit remains in its initial state.

The initial state is indicated with a ket on the left

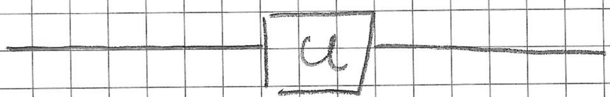


The number of qubits prepared in that state, is indicated with a slash and number n



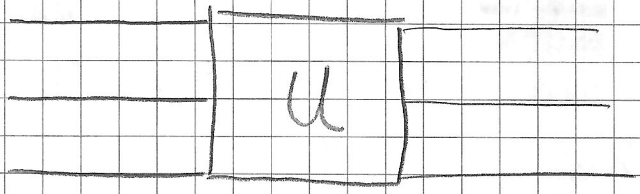
Quantum operators

A single qubit operator is denoted as



A binary gate (double qubit operator) as

A ternary gate looks like



Some operators have special designation (usually those involved in universality)

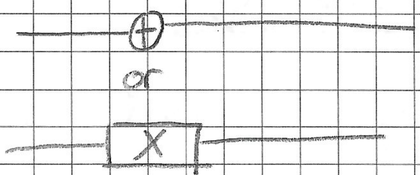
Common Unary Operators

Unitary operations allow interaction with and manipulation in a manner that allows computation

There are uncountably many unitary operators, but there is a distinct set that is Turing complete which is most commonly used in QCs today.

The Pauli group (after Wolfgang Pauli) consists of:

$$X := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$



This acts like a NOT operator

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0+0 \\ 1+0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle$$

the representation of the operator is

$$X := |0\rangle\langle 1| + |1\rangle\langle 0|$$

the application of the X operator is represented as

$$X|j\rangle = |j \oplus 1\rangle \qquad j \in \{0,1\}$$

$\oplus$ is addition modulo-2 hence this is a NOT operator

$$|0\rangle \;—\!—\!\oplus\!—\!—\; |1\rangle$$

$$Y := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad —\boxed{Y}—$$

Y operator (also denoted as σ_y) rotates the state about y-axis

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0-i \\ 0+0 \end{pmatrix} = \begin{pmatrix} -i \\ 0 \end{pmatrix} = -i|0\rangle$$

$$Z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad —\boxed{Z}—$$

Z operator (also denoted as σ_z) rotates the state about z-axis

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1+0 \\ 0+0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = (-1)^0 |0\rangle = |0\rangle$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0+0 \\ 0-1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} = (-1)^1 |1\rangle = -|1\rangle$$

$$Z|j\rangle = (-1)^j |j\rangle$$

There is an another gate commonly introduced along the three Pauli gates above

$$I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

This is called the "identity gate" "memory gate" or "delay gate"

The reason why they are introduced together is that Pauli matrices + Identity matrix form a group P_1 which obeys

$$P_a P_b = \underbrace{\delta_{ab}}_{\text{Kronecker delta}} I + i \underbrace{\epsilon_{abc}}_{\text{Levi-Civita symbol}} P_c$$

P_1 span the single qubit operator space, therefore any single-qubit operation can be decomposed into a linear combination of the identity operator and the Pauli operators

The general phase shift operator is

$$R_\varphi := \begin{pmatrix} 1 & 0 \\ 0 & e^{i\varphi} \end{pmatrix}$$

Two additional rotations of significance apart from the Paulis are

$S := \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$ —$\boxed{S}$— $T := \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$ —$\boxed{T}$—

$\varphi = \pi/2$ $\varphi = \pi/4$

Note that $S = T^2$

The crucial single qubit operator for Quantum Computing is called the Hadamard operator

$$H := \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad —\boxed{H}—$$

– (Actually John Sylvester developed this, but it is named after Jacques Hadamard, see Stigler's law of eponymy)

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1+0 \\ 1+0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0+1 \\ 0-1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

H takes a computational basis state $|0\rangle$ or $|1\rangle$ and puts it into a Hadamard state $|+\rangle, |-\rangle$

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \qquad |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

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then

$$H = |+\rangle\langle 0| + |-\rangle\langle 1|$$

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{\sqrt{2}}(X+Z)$$

since Hadamard is Hermitean $HH = I$ hence

$$H = |0\rangle\langle +| + |1\rangle\langle -|$$

i.e. it also takes a Hadamard state and maps it into a computational state

The eigenstates of single qubit Paulis are

	+1 eigenstate	-1 eigenstate
I	all	none
X	$\|+\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$\|-\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$
Y	$\|+i\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix}$	$\|-i\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix}$
Z	$\|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

For any unitary gate that has the action

$U(\alpha|0\rangle + \beta|1\rangle) = \alpha'|0\rangle + \beta'|1\rangle$, then $HUH(\alpha|+\rangle + \beta|-\rangle)$ $= \alpha'|+\rangle + \beta'|-\rangle$.

Note that matrix representations are defined up to a global phase, and they add a global phase to the eigenstates of their respective states.

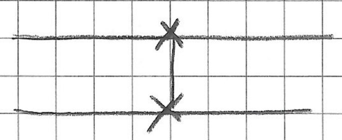
An interpretation of rotation gates and the origin of their name can be found in Bloch sphere visualisation

Binary operators

In two-qubit or binary operators, following computational basis is used

$$|00\rangle = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \quad |01\rangle = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \quad |10\rangle = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \quad |11\rangle = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

$$\text{SWAP} := \begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix}$$

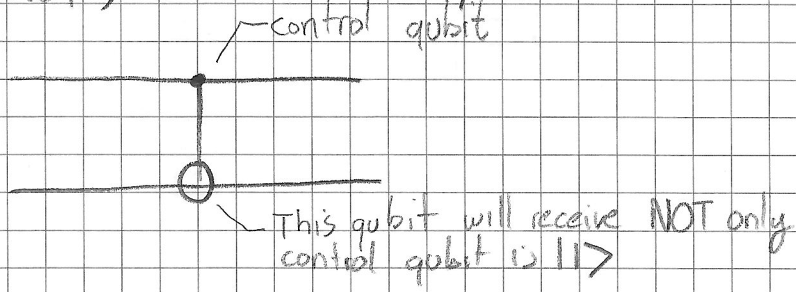


(two qubits are swapped)

SWAP takes state $|01\rangle$ to $|10\rangle$ and $|10\rangle$ to $|01\rangle$

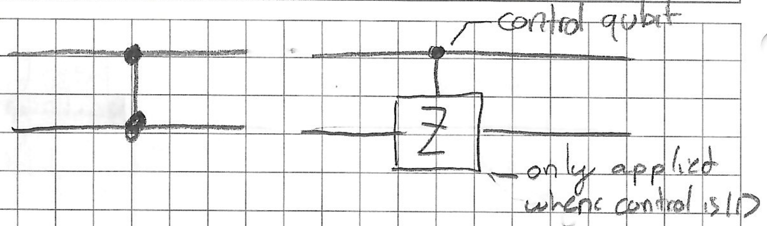
A critical gate for quantum computing is controlled-NOT (CNOT)

$$\text{CNOT} := \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}$$



Notice $\text{SWAP}_{ij} = \text{CNOT}_{ij}\, \text{CNOT}_{ji}\, \text{CNOT}_{ij}$

$$CZ := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$



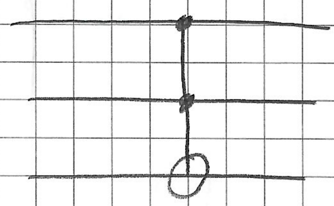
unlike CNOT gate, CZ gate is symmetric, either qubit can be the control qubit

Ternary operators

Toffoli operator (CCNOT): Both control qubits must be $|1\rangle$ to apply NOT to third qubit.

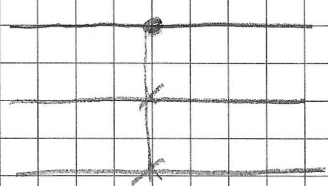
$$(x, y, z) \longmapsto (x, y, (z \oplus xy))$$

$$CCNOT := \begin{pmatrix} 1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&1&0 \end{pmatrix}$$



Fredkin or CSWAP gate applies SWAP operation to last two qubits controlled by the first qubit

$$CSWAP := \begin{pmatrix} 1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&1 \end{pmatrix}$$



Universality of Quantum Operators

In classical computing, NAND gate alone is sufficient to construct all the gates required to make the computer "Turing complete" or "universal"

In Quantum Computing there are several combinations of unary and binary operators that lead to universality

1.) Toffoli gate when paired with a basis changing unary operator (such as Hadamard)

2) {CNOT, T, H}

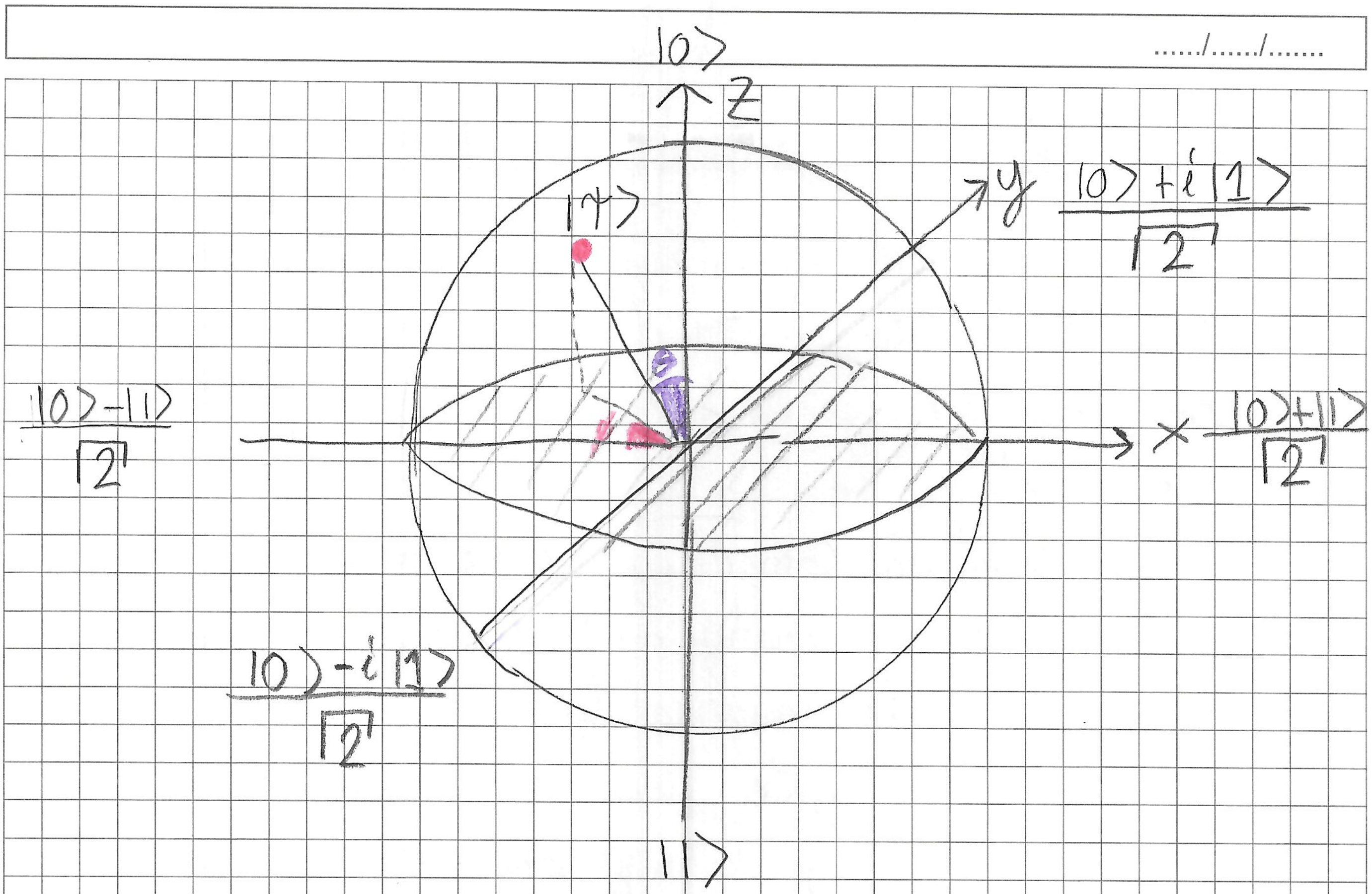
Bloch Sphere Representation

Any state in a quantum computation can be represented as a vector that begins at the origin and terminates on the surface of the unit Bloch sphere.

As a convention, two antipodes are $|0\rangle$ on top and $|1\rangle$ on the bottom

Different phases can be represented along X and Y axes

Interpretation of Universality: A set of operators that enable us to reach any point in the Bloch sphere



The Bloch Sphere