Phys 213 Final Examination 10/01/2023 17:00-19:00

Proctor's remark:				Initials	
Full Name		Student ID		Signature	

Problem 1 of 4 (20 pts) Answers without solution steps clearly shown will not be given any credit.

The specific heat of water is c_w and its density is ρ_w. You have V_H volume of hot water at T_H and you are using a cold lake of volume V_C temperature T_C as sink. Calculate the maximum useful work that can be extracted using the concept of entropy $dS = dQ/T$. Note that the change in the temperature of T_H is **not** small.

a) (5 pts) Consider you use a heat transfer process to bring the temperature of the hot water and the lake from an arbitrary initial low temperature T_i to T_H and T_i to T_C using quasistatic processes separately. Use the concept of specific heat, $dQ = McdT$, to calculate the total entropy change in the hot water and lake system.

$$\Delta S_{\text{hot water}} = \int_{T_i}^{T_H} \frac{dQ}{T} = \rho_w V_H c_w \int_{T_i}^{T_H} \frac{dT}{T} = \rho_w V_H c_w \ln \frac{T_H}{T_i} \qquad \Delta S_{lake} = \rho_w V_c c_w \ln \frac{T_c}{T_i}$$

$$\Delta S = \Delta S_{\text{hot water}} + \Delta S_{\text{lake}} = \boxed{\rho_w c_w \left(V_H \ln \frac{T_H}{T_i} + V_c \ln \frac{T_c}{T_i}\right)}$$

b) (5 pts) Now consider you use a heat transfer process to bring the temperature of the lake+hot water from an arbitrary initial low temperature T_i to $T_C + \delta$

$$\Delta S_{\substack{lake\\+\\hot}} = \int_{T_i}^{T_c+\delta} \frac{dQ}{T} = c_w \rho_w (V_c + V_H) \int_{T_i}^{T_c+\delta} \frac{dT}{T} = \rho_w (V_c + V_H) \ln \frac{T_c+\delta}{T_i} c_w$$

$$\Delta S = \Delta S_{\text{hot water+lake}} = \boxed{c_w \rho_w (V_c + V_H) \ln \frac{T_c+\delta}{T_i}}$$

c) (5 pts) Calculate δ if the conditions for extracting maximum useful work is satisfied in the change of entropy. (tip:$(M_H + M_C)c_w \ln(T_C + \delta) \approx (M_H + M_C)c_w \ln T_C + \frac{(M_H+M_C)c_w\delta}{T_C}$ where M_H is the mass of the hot water and M_C is the mass of the lake)

Maximum useful work → reversible process → entropy is the same for a) and b) (nothing is lost to "friction"))

$$\cancel{\rho_w} \cancel{c_w} \left(V_H \ln \frac{T_H}{\cancel{T_i}} + V_c \ln \frac{T_c}{\cancel{T_i}}\right) = \cancel{\rho_w} (V_c + V_H) \ln \frac{T_c+\delta}{\cancel{T_i}} \cancel{c_w}$$

use tip

$$(\cancel{V_c} + V_H) \ln T_c + \frac{(V_c+V_H)\delta}{T_c} = V_H \ln T_H + \cancel{V_c \ln T_c}$$

$$\delta = \frac{T_c}{V_c+V_H} (V_H \ln T_H - V_H \ln T_c)$$

$$\delta = \boxed{\frac{V_H}{V_c+V_H} T_c \ln \frac{T_H}{T_c}}$$

d) (5 pts) Use the first law of thermodynamics ($\Delta E = Q - W$) to calculate the <u>maximum useful work</u> that can be extracted.

$$Q_H = W + Q_L \qquad \Delta E = 0 \text{ (reversible)}$$

$$W = \rho_w V_H c_w T_H + \rho_w V_c c_w T_c - (V_H + V_c)\rho_w c_w (T_c + \delta)$$

$$= \rho_w c_w \left(V_H T_H + \cancel{V_c T_c} - V_H T_c - \cancel{V_c T_c} - \frac{\cancel{(V_H+V_c)} V_H}{\cancel{(V_H+V_c)}} T_c \ln \frac{T_H}{T_c}\right)$$

$$= \rho_w c_w V_H \left(T_H - T_c - T_c \ln \frac{T_H}{T_c}\right)$$

$$W = \boxed{\rho_w c_w V_H \left(T_H - T_c - T_c \ln \frac{T_H}{T_c}\right)}$$

Problem 2 of 4 (20 pts) Answers without solution steps clearly shown will not be given any credit.

a) (10 pts) Consider two successive Lorentz transformations of the three frames of reference K_0, K_1, K_2. K_1 moves parallel to the x axis of K_0 with velocity v, and K_2 moves parallel to the x axis of K_1 with speed v. Given an object moving in the x direction with speed v_2 in K_2, derive the formula for the transformation of its velocity from K_2 to K_0 in terms of $\beta_0 = v_0/c$, $\beta_2 = v_2/c$ and $\beta = v/c$.

$$V_1 = \frac{V_2 + V}{1 + V_2 V/c^2} \rightarrow \beta_1 = \frac{\beta_2 + \beta}{1 + \beta_2 \beta} \quad \text{also} \quad \beta_0 = \frac{\beta_1 + \beta}{1 + \beta_1 \beta} = \frac{\frac{\beta_2 + \beta}{1+\beta_2\beta} + \beta}{1 + \frac{\beta_2 + \beta}{1+\beta_2\beta}\beta}$$

$$= \frac{\beta_2 + \beta + \beta + \beta_2 \beta^2}{1 + \beta_2\beta + \beta_2\beta + \beta^2} = \frac{\beta_2(1+\beta^2) + 2\beta}{2\beta_2\beta + (1+\beta^2)}$$

$$\beta_0 = \boxed{\frac{\beta_2 + 2\beta/(1+\beta^2)}{1 + 2\beta_2\beta/(1+\beta^2)}}$$

b) (10 pts) Now generalize a) to $n+1$ frames moving with the same speed v along x relative to one another. Derive the formula for a Lorentz transformation from K_n to K_0 if the speed of the object in K_n is also v using the concept of "rapidity" ($\beta_i = \tanh w_i$) and the relationship

$$\tanh(w_1 + w_2) = \frac{\tanh w_1 + \tanh w_2}{1 + \tanh w_1 \tanh w_2}$$

β_2 to β_0 $\rightarrow$

$$\beta_1 = \frac{\beta_2 + \beta}{1 + \beta_2\beta} = \frac{\tanh w_2 + \tanh w}{1 + \tanh w_2 \tanh w} = \tanh(w_2 + w) = \tanh(w_1)$$

$$\beta_0 = \frac{\tanh w_1 + \tanh w}{1 + \tanh w_1 \tanh w} = \tanh(w_1 + w) = \tanh(\underbrace{w_2}_{w} + w + w)$$

$(n+1)w$

generalise $\rightarrow$ $\beta_0 = \tanh(w_n + w_{n-1} + \cdots + w_1 + w) = \tanh[(n+1)\tanh^{-1}\beta]$

$$\beta_0 = \boxed{\tanh[(n+1)\tanh^{-1}\beta]}$$

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Problem 3 of 4 (30 pts) Answers without solution steps clearly shown will not be given any credit.

a) (10 pts) Write down the time independent Schrödinger equation for simple harmonic oscillator where a restoring force $F = -Cx$ acts on an object with mass m resulting in a potential of $U(x) = \frac{1}{2}Cx^2$.

$$\boxed{-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2}Cx^2\psi = E\psi}$$

b) (20 pts) Use the solution to the above equation $\psi_n = \frac{1}{\sqrt{2^n n!}}\frac{m\omega}{\pi\hbar}e^{-Bx^2}H_n(\sqrt{2B}x)$ where H_n are the physicists' Hermite polynomials $H_n(z) = (-1)^n e^{z^2}\frac{d^n}{dz^n}(e^{-z^2})$ and $\omega = \sqrt{C/m}$ to calculate the ground state energy (n=0) in terms of ω (tip: what should be B to satisfy the Schrödinger equation?). What is the difference between the ground state of quantum harmonic oscillator and the classical harmonic oscillator?

$$H_0 = (-1)^0 e^{z^2} e^{-z^2} = 1 \qquad \frac{d^2\psi_0}{dx^2} = 2(2Bx^2-1)ABe^{-Bx^2}$$

$$\psi_0 = \underbrace{\frac{1}{\sqrt{2}}\frac{m\omega}{\pi\hbar}}_{A} e^{-Bx^2}$$

$$-\frac{\hbar^2}{2m}\left[2(2Bx^2-1)\cancel{A}Be^{-Bx^2}\right] + \frac{1}{2}Cx^2\cancel{A}e^{-Bx^2} = E_0\cancel{A}e^{-Bx^2}$$

$$\left(\frac{\hbar^2 B}{m} - E_0\right) + \left(\frac{1}{2}C - \frac{2\hbar^2B^2}{m}\right)x^2 = 0$$

$$E_0 = \frac{\hbar^2 B}{m} \qquad \frac{1}{2}C = \frac{2\hbar^2B^2}{m} \rightarrow B = \frac{\sqrt{mC}}{2\hbar} = \frac{m}{2\hbar}\sqrt{\frac{C}{m}} = \frac{\omega m}{2\hbar}$$

$$E_0 = \frac{\hbar^2}{m}\frac{\sqrt{mC}}{2\hbar} = \frac{1}{2}\hbar\sqrt{\frac{C}{m}} = \frac{1}{2}\hbar\omega$$

Classical harmonic oscillator has minimum energy 0

$$E_0 = \boxed{\frac{1}{2}\hbar\omega}$$

$$E_0 - E_{\text{classical}} = \boxed{\frac{1}{2}\hbar\omega}$$

Problem 4 of 4 (30 pts) Answers without solution steps clearly shown will not be given any credit.

A particle of mass m is contained in a one-dimensional <u>impenetrable</u> box extending from $x = -L/2$ to $x = +L/2$. The particle is in its ground state.

a) (10 pts) Find the eigenfunctions of the ground state and the first excited state (notice x is **not starting at 0)**

$$\psi(x) = A\sin kx + B\cos kx$$

$$\psi\left(-\tfrac{L}{2}\right) = 0 \rightarrow A\sin\left(-\tfrac{kL}{2}\right) + B\cos\left(-\tfrac{kl}{2}\right) = 0$$

$$+\ \psi\left(\tfrac{L}{2}\right) = 0 \rightarrow A\sin\left(\tfrac{kl}{2}\right) + B\cos\left(\tfrac{kl}{2}\right) = 0$$

$$\cos\left(\tfrac{kl}{2}\right) = 0 \rightarrow k = \tfrac{n\pi}{l}$$

$$1 = |\psi_0|^2 = B^2\int_{-l/2}^{l/2}\cos^2\left(\tfrac{\pi}{l}x\right)dx = \frac{B^2 l}{2\pi}\int_{-\pi/2}^{\pi/2}(1-\sin 2\theta)\,d\theta = \frac{B^2 l}{2} \qquad B = \sqrt{\tfrac{2}{l}}$$

$$|\psi_1|^2 = 1 = A^2\int_{-l/2}^{l/2}\sin^2\left(\tfrac{2\pi}{l}x\right)dx \Rightarrow A = \sqrt{\tfrac{2}{l}}$$

$$\psi_0 = A\sin\left(\tfrac{\pi}{l}x\right) + B\cos\left(\tfrac{\pi}{l}x\right)$$

$$\psi_0(l/2) = 0 \rightarrow A = 0 \ (\text{since } \sin\pi/2 \neq 0)$$

$$\psi_1 = A\sin\left(\tfrac{2\pi}{l}x\right) + B\cos\left(\tfrac{2\pi}{l}x\right)$$

$$\psi_1(l/2) = 0 \rightarrow B = 0 \ (\text{since } \cos\pi \neq 0)$$

$$\psi_0 = \boxed{\sqrt{\tfrac{2}{L}}\cos\tfrac{\pi x}{L}}$$

$$\psi_1 = \boxed{\sqrt{\tfrac{2}{L}}\sin\tfrac{2\pi x}{L}}$$

b) (10 pts) The walls of the box are moved outward instantaneously to form a box extending from $x = -L$ to $x = +L$. Calculate the probability that the particle will stay in the ground state during this sudden expansion. (tip: the probability of transition from wavefunctions can be calculated as $P_{0j} = |I_{0j}|^2$ where $I_{0j} = \int_a^b \psi_j'(x)\psi_0(x)dx$)

$$\tfrac{L}{2} \rightarrow L \qquad \psi_0'(x) = \frac{1}{\sqrt{L}}\cos\frac{\pi x}{2L} \qquad \psi_1'(x) = \frac{1}{\sqrt{L}}\sin\frac{\pi x}{L}$$

$$I_{00} = \frac{\sqrt{2}}{L}\int_{-L/2}^{L/2}dx\,\cos\frac{\pi x}{2L}\cos\frac{\pi x}{L} = \frac{1}{\sqrt{2}L}\int_{-L/2}^{L/2}dx\left(\cos\frac{\pi x}{2L} + \cos\frac{3\pi x}{2L}\right)$$

$$= \frac{1}{\sqrt{2}\cancel{L}}\frac{4\cancel{L}}{\pi}\left(\sin\frac{\pi}{2\cancel{L}}\frac{\cancel{L}}{2} + \frac{1}{3}\sin\frac{3\pi}{2\cancel{L}}\frac{\cancel{L}}{2}\right) = \frac{1}{\sqrt{2}}\frac{4}{3\pi}\left(\sin\tfrac{\pi}{4} + \sin\tfrac{3\pi}{4}\right) = \frac{8}{3\pi}$$

$$P_{00} = \boxed{(8/3\pi)^2}$$

c) (10 pts) Calculate the probability that the particle jumps from the initial ground state to the first excited final state.

$$I_{01} = \frac{\sqrt{2}}{L}\int_{-L/2}^{L/2}dx\,\sin\frac{\pi x}{L}\cos\frac{\pi x}{L} = 0 \ (\text{parity})$$

$$P_{01} = \boxed{0}$$