Phys 213 Make-up Examination 11/01/2023 17:00-19:00

| Proctor's remark: | | | | Initials | |
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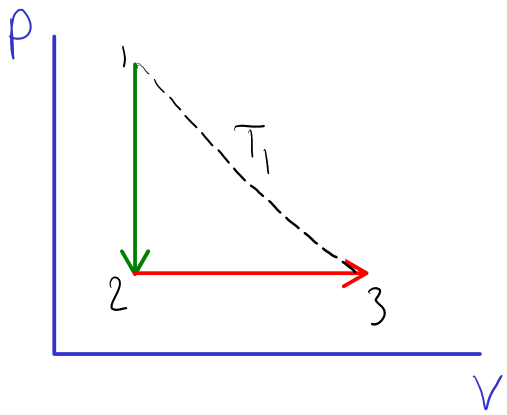
| Full Name | | Student ID | | Signature | |

**Problem 1 of 4** (20 pts) Answers without solution steps clearly shown will not be given any credit.

An ideal gas of total mass $m$ and molecular weight $\mu$ is isochorically (at constant volume) cooled to a pressure $n$ times smaller than the initial pressure $P_1$. The gas is then expanded at constant pressure so that in the final state the temperature $T_2$ coincides with the initial temperature $T_1$.

a) (10 pts) Draw PV diagram of this process. Identify the section responsible for the work done.




$W = \int P\,dV$

only possible in $2 \rightarrow 3$

b) (10 pts) Calculate the work done by the gas.

$$W_{2-3} = \int_2^3 P\,dV = P_2(V_2 - V_1) = P_2 V_2 \left(1 - \frac{V_1}{V_2}\right)$$

$$P_1 V_1 = \frac{m}{\mu} R T_1 \qquad P_2 V_2 = \frac{m}{\mu} R T_2 = \frac{m}{\mu} R T_1$$

Since initial and final temperatures are the same

$$W = \frac{m}{\mu} R T_1 \left(1 - \frac{P_2}{P_1}\right) = \frac{m}{\mu} R T_1 \left(1 - \frac{1}{n}\right)$$

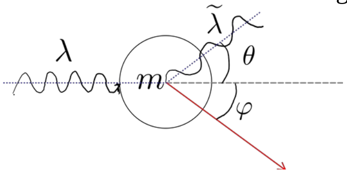
$W = \boxed{\frac{m}{\mu} R T \left(1 - \frac{1}{n}\right)}$

Phys 213 Make-up Examination 11/01/2023 17:00-19:00

**Problem 2 of 4** (20 pts) Answers without solution steps clearly shown will not be given any credit.

In the Compton effect, a $\gamma$-ray photon of wavelength $\lambda$ strikes a free, but initially stationary, electron of mass $m$. The photon is scattered an angle $\theta$ and its scattered wavelength is $\widetilde{\lambda}$ . The electron recoils at an angle $\varphi$

$$\mathcal{E} = \sqrt{p^2c^2 + m^2c^4}$$
$$\mathcal{E}_{\text{photon}} = pc$$
$$\mathcal{E}_{\text{rest}} = mc^2$$




a) (10 pts) Write the relativistic equations for momentum and energy conservation (using $\vec{p}, \widetilde{\vec{p}}, \mathcal{E}, \widetilde{\mathcal{E}}$ are the momenta and energies of the photon before and after the scattering, $\vec{p}_e, \widetilde{\vec{p}}_e, \mathcal{E}_e, \widetilde{\mathcal{E}}_e$, are the momenta and energies of the electron before and after the scattering)

$$\vec{P}_e = 0$$
$$\mathcal{E} + \mathcal{E}_e = \tilde{\mathcal{E}} + \tilde{\mathcal{E}}_e$$
$$\mathcal{E}_e = mc^2 \qquad \tilde{\mathcal{E}}_e = \sqrt{\tilde{P}_e c^2 + m^2c^4}$$
$$\mathcal{E} = pc \qquad \tilde{\mathcal{E}} = \tilde{p}c$$

Momentum: $\boxed{\vec{p} - \tilde{\vec{p}} = \tilde{\vec{P}}_e}$

Energy: $\boxed{pc + mc^2 = \tilde{p}c + \sqrt{\tilde{P}_e^{\,2} c^2 + m^2c^4}}$

b) (10 pts) Find an expression for the change $\widetilde{\lambda} - \lambda$ in the photon wavelength for the special case $\theta = \pi/2$ (use $p = h/\lambda$)

① $\tilde{\vec{P}}_e^{\,2} = (\vec{p} - \tilde{\vec{p}})^2$ (from momentum conservation)

$(pc + mc^2 - \tilde{p}c)^2 = \tilde{P}_e^{\,2} c^2 + m^2c^4$ (from energy conservation)

$\cancel{c^2}((p-\tilde{p}) + mc)^2 + \cancel{c^2} mc^2 = \tilde{P}_e^{\,2} \cancel{c^2}$

② $\tilde{P}_e^{\,2} = (p-\tilde{p})^2 + 2mc(p-\tilde{p})$

① $\rightarrow$ $\tilde{P}_e^{\,2} = \tilde{\vec{P}}_e^{\,2} = \vec{p}\cdot\vec{p} + \tilde{\vec{p}}\cdot\tilde{\vec{p}} - 2\vec{p}\cdot\tilde{\vec{p}} = pp + \tilde{p}\tilde{p} - 2p\tilde{p}\cos\theta$

use ② $\longrightarrow$ $\cancel{pp} + \cancel{\tilde{p}\tilde{p}} - \cancel{2}p\tilde{p} + \cancel{2}mc(p-\tilde{p}) = \cancel{pp} + \cancel{\tilde{p}\tilde{p}} - \cancel{2}p\tilde{p}\cos\theta$

$p\tilde{p}(1-\cos\theta) = mc(p-\tilde{p})$

$\theta = \pi/2 \qquad \cos\theta = 0$

$\frac{p\tilde{p}}{p\tilde{p}} = \frac{mc(p-\tilde{p})}{p\tilde{p}} \longrightarrow 1 = mc\left(\frac{1}{\tilde{p}} - \frac{1}{p}\right)$ ($h/\tilde{\lambda}$, $h/\lambda$)

$mc\left(\frac{\tilde{\lambda}}{h} - \frac{\lambda}{h}\right) = 1$

$\widetilde{\lambda} - \lambda = \boxed{h/mc}$

Phys 213 Make-up Examination 11/01/2023 17:00-19:00

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**Problem 3 of 4** (30 pts) Answers without solution steps clearly shown will not be given any credit.

An electron is confined to one dimension to the right half-space $x > 0$ outside a perfect conductor.

a) (10 pts) The image potential of an electron outside a perfect conductor is $U(x) = -\dfrac{e^2}{4x}$. Write down the time independent Schrödinger equation for this electron.

$$\boxed{-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} - \frac{e^2}{4x}\psi(x) = E\psi(x)}$$

b) (20 pts) Using $\psi = Nxe^{-\alpha x}$, Find the ground state energy and the normalization constant $N$ $\int_0^\infty x^2 e^{-2\alpha x}\,dx = \dfrac{1}{4\alpha^3}$.

$$\frac{d^2\psi(x)}{dx^2} = N\frac{d}{dx}\left(e^{-\alpha x} - \alpha x e^{-\alpha x}\right) = N\left(-\alpha e^{-\alpha x} - \alpha e^{-\alpha x} + \alpha^2 x e^{-\alpha x}\right)$$

$$= N\alpha(\alpha x - 2)e^{-\alpha x}$$

$$\frac{\hbar^2}{2m}\cancel{N}\alpha(2-\alpha x)\cancel{e^{-\alpha x}} - \frac{e^2}{4\cancel{x}}\cancel{N}\cancel{x}\cancel{e^{-\alpha x}} = E\cancel{N}x\cancel{e^{-\alpha x}}$$

$$\overbrace{\left(\frac{\hbar^2\alpha}{m} - \frac{e^2}{4}\right)}^{0} + \overbrace{\left(-\frac{\hbar^2\alpha^2}{2m} - E\right)}^{0}x = 0$$

$$\alpha = \frac{e^2 m}{4\hbar^2} \qquad E = -\frac{\hbar^2\alpha^2}{2m} = -\frac{e^4 m}{32\hbar^2}$$

Side note: $\alpha = 1/4a_0$ ($a_0$: Bohr radius) $E = -\dfrac{E_R}{16}$ → ground state energy of Hydrogen atom / Rydberg

$$1 = N^2\int_0^\infty dx\, x^2 e^{-2\alpha x} = \frac{N^2}{4\alpha^3}$$

$$N = \sqrt{4\alpha^3}$$

$$E_0 = \boxed{-\frac{e^4 m}{32\hbar^2}}$$

$$N = \boxed{\sqrt{4\alpha^3}}$$

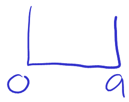
Phys 213 Make-up Examination 11/01/2023 17:00-19:00

**Problem 4 of 4** (30 pts) Answers without solution steps clearly shown will not be given any credit.

Two electrons are confined in one dimension to a infinite potential box starting from $x = 0$ to $x = a$ . A clever experimentalist has arranged that both electrons have the same spin state. Ignore the Coulomb interaction between electrons.

a) (15 pts) Find the orbital part of the ground state wave function for the two-electron system by identifying $A, b, c, \psi_1, \psi_2$ in $\psi(x_1, x_2) = A\,[\cancel{a}\,b\,\psi_1(x_1)\psi_2(x_2) + \cancel{b}\,c\,\psi_1(x_2)\psi_2(x_1)]$ where $\psi(x)$ are normalized "particle in a box" solutions for each electron individually and considering the fact that the overall electronic wavefunction must be antisymmetric in an exchange operation $\psi(x_1, x_2) = -\psi(x_2, x_1)$ (since they are fermions).

$$\psi(x) = \sqrt{\frac{2}{a}} \sin \frac{n\pi x}{a}$$




$$A[b\,\psi_1(x_1)\psi_2(x_2) + c\,\psi_1(x_2)\psi_2(x_1)] = -A[b\,\psi_1(x_2)\psi_2(x_1) + c\,\psi_1(x_1)\psi_2(x_2)]$$

$b = -c$ set $b = 1$ (since $A$ will normalize)

notice $\psi_1$ can not be equal to $\psi_2$ to satisfy this so for ground state

$$\psi_0(x_1, x_2) = A\left[\frac{2}{a}\sin\frac{\pi x_1}{a}\sin\frac{2\pi x_2}{a} - \frac{2}{a}\sin\frac{\pi x_2}{a}\sin\frac{2\pi x_1}{a}\right]$$

$$A^2(1 + 1 + 0 + 0) = 1$$

$$A = \frac{1}{\sqrt{2}}$$

$$\psi_0(x_1, x_2) = \boxed{\frac{1}{\sqrt{2}}\left(\frac{2}{a}\sin\frac{\pi x_1}{a}\sin\frac{2\pi x_2}{a} - \frac{2}{a}\sin\frac{\pi x_2}{a}\sin\frac{2\pi x_1}{a}\right)}$$

b) (15 pts) Calculate the probability that both electrons are on the left side ($0 < x < a/2$) of the box. $P_{\mathrm{LL}} = \int_0^{a/2} dx_1 \int_0^{a/2} dx_2\ |\psi_0(x_1, x_2)|^2$

$$P_{LL} = \int_0^{a/2} dx_1 \int_0^{a/2} dx_2\, [\psi_1^2(x_1)\psi_2^2(x_2) - \psi_1(x_1)\psi_2(x_2)\psi_1(x_2)\psi_2(x_1)]$$

$$= I_1 I_2 - I_{12}^2$$

$$I_1 = \frac{2}{a}\int_0^{a/2} dx\, \sin^2\frac{\pi x}{a} = 1/2$$

$$I_2 = \frac{2}{a}\int_0^{a/2} dx\, \sin^2\frac{2\pi x}{a} = 1/2$$

$$I_{12} = \frac{2}{a}\int_0^{a/2} dx\, \sin\frac{\pi x}{a}\sin\frac{2\pi x}{a} = \frac{4}{a}\int_0^{a/2} dx\, \sin^2\frac{\pi x}{a}\cos\frac{\pi x}{a}$$

$$= \frac{4}{3\pi}\sin^3\frac{\pi x}{a}\Big|_0^{a/2} = \frac{4}{3\pi}$$

$$P_{LL} = \left(\frac{1}{2}\right)^2 - \left(\frac{4}{3\pi}\right)^2$$

$$P_{\mathrm{LL}} = \boxed{\frac{1}{4} - \left(\frac{4}{3\pi}\right)^2}$$